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HOMOGENIZATION OF TRANSPORT EQUATIONS*

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Abstract. This article is devoted to the mathematical analysis of various formulas giving the equivalent absorption and scattering cross section for mixed materials in linear transport theory. We begin with a general result on the treatment of high-frequency oscillations in linear transport equations which is partly based upon the velocity averaging results and is the analogue, for transport equations, of the compensated compactness class of results. The case of periodic inhomogeneities is then studied in detail; in particular we show the essential difference with periodic homogenization of diffusion equations, due to small divisor problems. These results were announced in [F. Golse, *C.R. Acad. Sci. Paris Sér.* I *Math.*, 305 (1987), pp. 801–804; F. Golse, *Mathematical Aspects of Fluid and Plasma Dynamics*, G. Toscani, V. Boffi, and S. Rionero, eds., Lecture Notes in Math 1460, Springer-Verlag, Berlin, New York, 1991, pp. 152–169]. Finally, we treat a case of stochastic inhomogeneities in linear transport theory inspired from results due to Papanicolaou–Varadhan on the homogenization of diffusion processes.

Key words. homogenization, transport equations, periodic media, random media

AMS subject classifications. 35B27, 35B40, 35Q99, 35C20, 60F99, 73B35, 73B27

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1. Introduction.

The mathematical study of linear transport equations with fastly oscillating coefficients arises naturally in various contexts. For example, the classical averaging method for perturbations of ordinary differential equations, initiated by Bogoliubov–Mitropolski and studied in detail by several authors since then (see [3], [4], [27], and the references therein), can be recast in terms of linear partial differential equations of order 1; see, for example, [21] and [22] for a general discussion, and [17], where a proof of Anosov's very general averaging result is obtained in the case of vector fields having only Sobolev—and not Lipschitz—regularity, with the help of the DiPerna–Lions theory of generalized flows and renormalized solutions [13].

Here we shall not concentrate on the type of linear transport equations associated to vector fields (Liouville equations), but instead on linear transport equations with absorption and scattering terms modeling the collision of a population of particles with a background medium. Such equations occur naturally in nuclear engineering to model the flux of neutrons in a composite fissile material and in radiative transfer to model the flux of photons through a mixture of materials with different opacities. Such mixtures arise, for example, in the description of fusion pellets because of the various hydrodynamic instabilities (Richtmeyer–Meshkov, for example) developing near interfaces.

As in all asymptotic problems, the first step consists of the definition of the various scales relevant for the problem considered. We shall limit ourselves to the case where the problem can be characterized by only three length scales:

- L is the macroscopic length scale; it is given for example by the size of the domain where the transport equation is to be studied, or by the scale of

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variation of the initial data if the domain is infinite.

- l is a microscopic length scale characteristic of the structure of the inhomogeneities: it can be the average distance between two neighboring inhomogeneities or the average size of the inhomogeneities.
- λ is the mean free path of the particles in the material.

In most realistic problems, these three scales would not suffice. For example, the average size of the inhomogeneities may be of a very different order than the average distance between two neighboring inhomogeneities; we refer to [9] for a very simple yet fundamental example where such considerations indeed play a very important role. It may also be that the geometric structure of the inhomogeneities requires more than one length scale (if one thinks of filaments, for example). It could also be that the mean free path of the particles is very different in the various components of the mixture: this type of difficulty occurs most frequently in radiative transfer. Nevertheless, we shall limit our mathematical analysis to models involving only the three scales above.

The homogenization problem for transport phenomena has been much studied in the case where $\lambda << l << L$. (Here, the notation $a << b$ means that $a/b \to 0$ asymptotically.) In such a case, one can approximate first the solution of the transport equation by that of a diffusion equation (see, for example, [6] for a very simple account of this theory) and then apply the existing homogenization results for diffusion equations (see, for example, [7] to the approximating equation). This strategy has been developed notably in [8], [32] (we also refer to the literature indicated there). The homogenization problem for the transport equation itself has received much less attention; however, there are very interesting contributions by Levermore et al. [25] and Levermore, Pomraning, and Wong [26] in the case of some particular random media—they study mainly the case of random binary mixtures such that the lengths of the intersections of any given straight line with any component of the mixture are exponentially distributed; see also [36]. Finally, we mention two recent contributions on the spectral problem in the context of neutron transport theory [1], [2].

In the present paper, we shall restrict our attention to the case where $l << \lambda$ and we shall consider L and λ as not asymptotically small. In other words, the only small scale here will be l. Therefore, the only relevant model for such a situation is the transport equation itself, to which, as we shall see, the homogenization methods used on diffusion equations do not apply.

The outline of this paper is as follows:

- In section 2, we present a mathematical result which can be regarded as an analogue, in the case of kinetic models, of the compensated compactness theorems of Murat and Tartar (see [29], [33], [34]).
- In section 3, we show how this result can be applied to the most general result possible concerning the homogenization problem for the transport equation; this section is analogous to the study of H-convergence by Murat and Tartar on diffusion equations (see, for example, [28]).
- In section 4, we discuss the case of periodic inhomogeneities; we show in detail why the method used on diffusion equations does not apply here because of small divisor problems analogous to those encountered in the study of perturbations of dynamical systems (see, for example, [3] for a brief survey of small divisor problems and how they are usually dealt with).
- In section 5, we consider the homogenization problem for the transport equation in random media such as those studied by Papanicolaou–Varadhan [31] in the case of diffusion equations.

The results in sections 2 and 3 were announced in [19]; those in section 4 were described without proof in [20]. The homogenization problem for random media was announced in [14].

Finally, we quote some references on billiards dynamics (also known as the Lorentz gas model) which can be viewed as particular examples of homogenization problems for the transport equation (besides their intrinsic interest in statistical physics); see [23], [5], [15], [9], as well as the fundamental contributions by Bunimovich–Sinai [10] and Bunimovich, Sinai, and Chernov [11]. The analogous problem in the case of diffusion equations has been studied by Cioranescu–Murat [12].

2. Compensated compactness for kinetic models. In the homogenization problem for diffusion equations, a fundamental role is played by a series of results known as the compensated compactness method, due to Murat and Tartar [29], [33], and especially by the so-called "div-curl" lemma. The div-curl lemma as it is does not apply to kinetic models.

However, in [19], a result very much similar to the div-curl lemma in spirit and directly applicable to most kinetic models was announced [19, Theorem 1]. Since [19] provided only the idea of the proof, we shall in this section prove this result and discuss it in more detail.

PROPOSITION 2.1 (see [19]). *Let μ be a positive regular Borel measure on $\mathbf{R}^d$ such that, for any hypercube $P \subset \mathbf{R}^d$, $0 < \mu(P) < +\infty$. Let (a_n) and (b_n) be two sequences of measurable functions on $\mathbf{R}^m \times \mathbf{R}^d$ satisfying the following assumptions:*

$$a_n \to 0 \text{ and } b_n \to 0 \text{ in } L^\infty(\mathbf{R}^m \times \mathbf{R}^d, dxd\mu) \text{ weak-}*, \tag{2.1}$$

$$\forall \chi \in C_0^\infty(\mathbf{R}^m \times \mathbf{R}^d), \qquad \int_{\mathbf{R}^d} a_n(x,y)\chi(x,y)d\mu(y) \to 0 \quad \text{in } L^1(\mathbf{R}^m), \text{ as } n \to +\infty, \tag{2.2}$$

and $\forall\ R > 0$

$$\int_{|y|\le R}\int_{|x|\le R} \sup_{|z-y|\le\epsilon} |b_n(x,y) - b_n(x,z)|dxd\mu(y) \to 0 \tag{2.3}$$

uniformly in n as $\epsilon \to 0$. Then

$$a_n b_n dxd\mu \to 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^m \times \mathbf{R}^d). \tag{2.4}$$

Proof. Let $l \in \mathbf{N}^*$ and decompose the y-space as a disjoint union of hypercubes of side $1/l$: $\mathbf{R}^d = \bigcup_{i\in\mathbf{N}} P_i$. For $y \in \mathbf{R}^d$, denote by $i(y) \in \mathbf{N}$ the unique index i such that $y \in P_i$. For all $f \in L^1(\mathbf{R}^d, d\mu)$, we also denote by $\langle f\rangle_i$ the average of f over the hypercube P_i. Then

$$a_n b_n = a_n \sum_{i\ge 0} \langle b_n\rangle_i \mathbf{1}_{P_i} + a_n \left(b_n - \sum_{i\ge0} \langle b_n\rangle_i \mathbf{1}_{P_i} \right) \tag{2.5}$$

or, in other words,

$$a_n(x,y)b_n(x,y) = a_n(x,y)\langle b_n(x,\cdot)\rangle_{i(y)} + a_n(x,y)\left(b_n(x,y) - \langle b_n(x,\cdot)\rangle_{i(y)}\right). \tag{2.6}$$

Therefore, for any $\chi \in C_0^\infty(\mathbf{R}^m \times \mathbf{R}^d)$ with support in $B(0,R) \times B(0,R)$,

$$\left|\iint_{\mathbf{R}^m\times\mathbf{R}^d}\chi(x,y)a_n(x,y)\left(b_n(x,y)-\langle b_n(x,\cdot)\rangle_{i(y)}\right)dxd\mu(y)\right|$$

$$\leq \|a_n\|_{L^\infty}\cdot\|\chi\|_{L^\infty}\cdot\int_{|y|\leq R}\int_{|x|\leq R}\left|b_n(x,y)-\langle b_n(x,\cdot)\rangle_{i(y)}\right|dxd\mu(y)$$

$$\leq \|a_n\|_{L^\infty}\cdot\|\chi\|_{L^\infty}\cdot\int_{|y|\leq R}\int_{|x|\leq R}\sup_{|y-z|\leq\operatorname{diam}(P_i)}|b_n(x,y)-b_n(x,z)|dxd\mu(y)$$

$$\leq \|a_n\|_{L^\infty}\cdot\|\chi\|_{L^\infty}\cdot C_R\rho(l), \tag{2.7}$$

where C_R is a positive constant depending only on R and $\rho(l)\to 0$ as $l\to+\infty$ according to (2.3) above. Let $\epsilon>0$; we pick $l\in\mathbf{N}^*$ such that $\rho(l)<\epsilon$. With this choice of l, on the other hand,

$$\iint_{\mathbf{R}^m\times\mathbf{R}^d}\chi(x,y)a_n(x,y)\langle b_n(x,\cdot)\rangle_{i(y)}dxd\mu(y)$$

$$=\sum_{i\geq 0}\int_{\mathbf{R}^m}\langle b_n(x,\cdot)\rangle_i\int_{\mathbf{R}^d}a_n(x,y)\chi(x,y)\mathbf{1}_{P_i}(y)d\mu(y)dx\,. \tag{2.8}$$

The sum in (2.8) is finite since χ is supported in $B(0,R)\times B(0,R)$ and $B(0,R)$ is covered by a finite subfamily of $(P_i)_{i\in\mathbf{N}}$ (which is a family of disjoint hypercubes of side $1/l$). For all $i\in\mathbf{N}$,

$$\int_{\mathbf{R}^d}a_n(x,y)\chi(x,y)\mathbf{1}_{P_i}(y)d\mu(y)\to 0\quad\text{in } L^1(\mathbf{R}^m)\quad\text{as } n\to+\infty \tag{2.9}$$

according to (2.2). Indeed, (2.2) applies to the case with χ replaced by $\chi(x,y)\mathbf{1}_{P_i}(y)$ which is the limit in $L^1(\mathbf{R}^m\times\mathbf{R}^d,dxd\mu)$ of a sequence of C^∞ functions with compact support. Therefore, bringing together (2.7), (2.8), and (2.9),

$$\limsup_{n\to+\infty}\iint\chi(x,y)a_n(x,y)b_n(x,y)dxd\mu(y)\leq C_R\epsilon\|\chi\|_{L^\infty}\sup_{n\in\mathbf{N}}\|a_n\|_{L^\infty}\,, \tag{2.10}$$

which, since ϵ is arbitrary, proves (2.4). □

An important particular case is the following.

PROPOSITION 2.2. *Under the same assumptions as in Proposition* 2.1, *where* μ *is the Lebesgue measure on* $\mathbf{R}^d$ *and with* (2.3) *weakened in*

$$\forall R>0\,,\qquad \int_{|y|\leq R}\int_{|x|\leq R}|b_n(x,y)-b_n(x,y+w)|dxdy\to 0, \tag{2.3$'$}$$

uniformly in n *as* $|w|\to 0$, *the conclusion* (2.4) *holds.*

Remark. A practical way to check (2.3′) is the condition

$$\sup_n\int_{|x|\leq R}\|b_n(x,\cdot)\|_{W^{1,s}(B(0,R))}dx<+\infty \tag{2.3$''$}$$

for some $s > 0$.

Proof. Let ϕ be a nonnegative C^∞ function supported in the unit ball of $\mathbf{R}^d$ with integral and set $\phi_\epsilon(y) = \epsilon^{-d}\phi(\frac{y}{\epsilon})$. Define

$$b_{n,\epsilon}(x,y) = \int_{\mathbf{R}^d} b_n(x, y-z)\phi_\epsilon(z)dz\,. \tag{2.11}$$

First, one has

$$\|b_n - b_{n,\epsilon}\|_{L^1(B(0,R)\times B(0,R))}$$

$$\leq \int_{\mathbf{R}^d}\left(\int_{|x|\leq R}\int_{|y|\leq R}|b_n(x,y) - b_n(x,y-z)|dydx\right)\phi_\epsilon(z)dz \to 0 \tag{2.12}$$

as $\epsilon \to 0$, uniformly in n, according to (2.3′). Hence

$$\left|\iint_{\mathbf{R}^m\times\mathbf{R}^d}\chi(x,y)a_n(x,y)(b_n(x,y) - b_{n,\epsilon}(x,y))dxdy\right| \to 0 \tag{2.13}$$

as $\epsilon \to 0$, uniformly in n. On the other hand, one can decompose $b_{n,\epsilon}$ in Fourier modes in the variable y:

$$b_{n,\epsilon}(x,y) = (2\pi)^{-d}\int_{\mathbf{R}^d}\hat{b}_{n,\epsilon}(x,\eta)e^{i\eta\cdot y}d\eta \tag{2.14}$$

with, for all integers $k \geq 0$,

$$|\hat{b}_{n,\epsilon}(x,\eta)| \leq C_{\epsilon,k}(1+|\eta|)^{-k}\,. \tag{2.15}$$

Therefore, for any fixed ϵ, one has

$$\iint_{\mathbf{R}^m\times\mathbf{R}^d}\chi(x,y)a_n(x,y)b_{n,\epsilon}(x,y)dxdy$$

$$= (2\pi)^{-d}\iint_{\mathbf{R}^m\times\mathbf{R}^d}\hat{b}_{n,\epsilon}(x,\eta)\left(\int_{\mathbf{R}^d}\chi(x,y)a_n(x,y)e^{i\eta\cdot y}dy\right)dxd\eta \to 0 \tag{2.16}$$

by (2.15) and (2.2). Bringing together the convergence (2.16) (for each fixed ϵ) and the uniform (in n) convergence (2.13) gives the announced conclusion (2.4). ∎

As they are stated above, these results do not seem in themselves reminiscent of the div-curl lemma. However, at least in the case of kinetic models, the condition (2.2) on the sequence (a_n) is usually obtained by applying the method of velocity averaging. We recall it for the sake of being complete.

Proposition 2.3 (velocity averaging; see [24]). *Let μ be a positive regular Borel measure on $\mathbf{R}^d$ such that, for all hypercubes $P \subset \mathbf{R}^d$, $0 < \mu(P) < +\infty$, and*

$$\forall(\tau,\xi) \in \mathbf{R}\times\mathbf{R}^d\setminus\{0\}\,, \qquad \mu(\{y \in \mathbf{R}^d \,|\, \tau + y\cdot\xi = 0\}) = 0\,. \tag{2.17}$$

Let (a_n) be a sequence of $L^1(\mathbf{R}^{d+1}_{(t,x)}\times\mathbf{R}^d_y)$ such that

$$a_n \to 0\,, \qquad (\partial_t + y\cdot\nabla_x)a_n \to 0 \text{ weakly in } L^1(\mathbf{R}^{d+1}_{(t,x)}\times\mathbf{R}^d_y). \tag{2.18}$$

Then
(2.19)

$$\forall \chi \in C_0^\infty(\mathbf{R}^{d+1}\times\mathbf{R}^d), \quad \int_{\mathbf{R}^d} a_n(t,x,y)\chi(t,x,y)d\mu(y) \to 0 \textit{ strongly in } L^1(\mathbf{R}^{d+1}_{(t,x)} \times \mathbf{R}^d_y).$$

Conditions (2.3) and (2.3′) say that the oscillations in the sequence (b_n) come from the variable x only, while Proposition 2.3 shows that, whenever the images of the sequence (a_n) by the streaming operator are uniformly equally integrable, the y-averages (velocity averages) of a_n do not contain oscillations in the variable x. This situation is clearly analogous to the Murat–Tartar div-curl lemma. We refer to the formalism due to P. Gérard [18], encompassing both the compensated compactness results as well as the analogue of Proposition 2.2 obtained by replacing L^1 or L^∞ spaces by L^2 spaces; the main new tool in this work is the notion of microlocal defect measure independently introduced by Tartar [35].

3. The general homogenization result. We consider in this section the general homogenization problem for the linear transport equation

$$\partial_t f + v\cdot\nabla_x f + \sigma(x,v)f - \int_{\mathbf{R}^d} k(x,v,v')f(t,x,v')dv' = 0\,. \tag{3.1}$$

In the above equation, $f \equiv f(t,x,v)$ is the number of density particles which, at time t, are located at the position x and have velocity v. To avoid unessential complications, we shall assume that (3.1) is posed in the phase space defined by $(x,v) \in \mathbf{R}^d \times \mathbf{R}^d$ and we shall consider the Cauchy problem for (3.1) corresponding to the initial data

$$f(0,x,v) = f^{in}(x,v)\,, \qquad (x,v) \in \mathbf{R}^d \times \mathbf{R}^d\,. \tag{3.2}$$

The coefficient $\sigma \equiv \sigma(x,v)$ is the absorption cross section of the material, while the integral kernel $k \equiv k(x,v,v')$ is the local scattering cross section of the material. We refer the interested reader to [16] for a more thorough description of the physical situations described by this type of equation. As we said in the introduction, one should think of (3.1) as a model describing the advection of neutrons in some background material; the absorption and scattering of neutrons correspond to collisions of the neutrons with the atoms of the background material, potentially with creation of secondary neutrons by fission reactions. Suffice it to say that the material is represented by the functions σ and k.

The homogenization procedure is an approximation aimed at simplifying the computation of solutions of (3.1)–(3.2) in the case where the background material is an intimate mixture of two (or more) components. In what follows, as indicated in the introduction, we shall keep the macroscopic length scale L (given, for example, by the characteristic (spatial) length of variation of the initial data f^{in}) and the mean free path of particles λ (which is given by the order of magnitude of the reciprocal of σ multiplied by some average velocity of the population of particles considered) as fixed parameters. The only asymptotically small parameter is the microscopic length scale l corresponding to the size of the microstructure of the background material (for example, the average size of or distance between inhomogeneities). As l/L tends to 0 (which is the mathematical phrase for the asymptotic smallness condition $l << L$), the functions σ and k display very fast oscillations (corresponding to the different values of the cross sections in the elementary components of the mixture with smaller and smaller microstructures). These fast oscillations of the coefficients will induce fast oscillations on the solution of (3.1)–(3.2) itself. Fortunately one is usually not so much interested in the detail of the oscillations of the number density f, but rather in macroscopic averages of the number density. Hence, in order to solve numerically the problem (3.1)–(3.2), it is highly desirable to have a description of the macroscopic averages of the number density f in terms of macroscopic averages of the cross sections σ and k only.

Here we shall see that, in order to describe the macroscopic averages of the number density f, it suffices to know the local macroscopic averages of the cross sections σ and k. In the case of the diffusion problem, there is in general no such simple principle; for example, the case of laminated materials shows that one must know macroscopic averages of certain nonlinear functions of the diffusion matrix.

The mathematical formulation of this principle is as follows. Introduce the small parameter $\epsilon = l/L$; the various local macroscopic averages of the cross sections σ and k correspond to considering these cross sections as families indexed by ϵ, σ_ϵ, and k_ϵ, weakly converging for some L^p topology as $\epsilon \to 0$. This weak convergence of the coefficients in the problem (3.1)–(3.2) will induce weak compactness of the solution f_ϵ (which is now also a family indexed by ϵ). The homogenization problem reduces then to computing the weak limit of products of weakly converging families as $\epsilon \to 0$. This is why results like those in section 2 (or the compensated compactness theorems in the case of diffusion equations with oscillating diffusion matrices) are essential in order to solve the most general homogenization problem possible. Indeed, this approach, initiated by Murat and Tartar (see, for example, [28]), needs no particular assumption on the geometry of the microstructure of the background material, but just a consideration of the fact that the mixture is realized at a small enough scale.

THEOREM 3.1. *Let $\sigma_\epsilon \equiv \sigma_\epsilon(x,v)$ and $k_\epsilon \equiv k_\epsilon(x,v,v')$ be bounded families of $L^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ and $L^\infty(\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d)$, respectively, such that, as $\epsilon \to 0$,*

$$\sigma_\epsilon \to \sigma, \qquad k_\epsilon \to k \quad \text{in } L^\infty \text{ weak-}*. \tag{3.3}$$

Assume the existence of $V > 0$ such that

$$k_\epsilon(x,v,v') = 0 \quad \text{if } v' \notin B(0,V)\,, \tag{3.4}$$

as well as

$$\forall R > 0\,, \qquad \int_{|v|\le R}\int_{|x|\le R} |\sigma_\epsilon(x,v) - \sigma_\epsilon(x,v+w)|dxdv \to 0 \tag{3.5}$$

and

$$\forall R > 0\,, \qquad \int_{|v'|\le R}\int_{|v|\le R}\int_{|x|\le R} |k_\epsilon(x,v,v') - k_\epsilon(x,v,v'+w)|dxdvdv' \to 0, \tag{3.6}$$

uniformly in ϵ as $|w| \to 0$. Consider, for all $f^{in} \in L^\infty(\mathbf{R}^d \times \mathbf{R}^d)$, the solution f_ϵ of the Cauchy problem

$$\partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon + \sigma_\epsilon(x,v)f_\epsilon - \int_{\mathbf{R}^d} k_\epsilon(x,v,v')f_\epsilon(t,x,v')dv' = 0\,, \qquad x,v \in \mathbf{R}^d\,, \tag{3.7}$$

$$f(0,x,v) = f^{in}(x,v)\,, \qquad (x,v) \in \mathbf{R}^d \times \mathbf{R}^d\,. \tag{3.8}$$

Then, as $\epsilon \to 0$, $f_\epsilon \to f$ in $L^\infty(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}^d)$ weak-, where f is the solution of the Cauchy problem for*

$$\partial_t f + v\cdot\nabla_x f + \sigma(x,v)f - \int_{\mathbf{R}^d} k(x,v,v')f(t,x,v')dv' = 0\,, \qquad x,v \in \mathbf{R}^d\,, \tag{3.9}$$

with initial data (3.8).

Remark 3.2. In practice, one will replace conditions (3.5)–(3.6) by the assumption that there exists some $s > 0$ such that

$$\forall R > 0\,, \qquad \sup_{\epsilon} \int_{|x|\leq R} \|\sigma_\epsilon(x,\cdot)\|_{W^{1,s}(B(0,R))} dx < +\infty \tag{3.5$'$}$$

and

$$\forall R > 0\,, \qquad \sup_{\epsilon} \int_{|v|\leq R} \int_{|x|\leq R} \|k_\epsilon(x,v,\cdot)\|_{W^{1,s}(B(0,R))} dx dv < +\infty\,. \tag{3.6$'$}$$

A particularly trivial (but useful) example is the following case, which is important for applications:

$$\sigma_\epsilon \equiv \sigma_\epsilon(x)\,, \qquad k_\epsilon(x,v,v') = \kappa_\epsilon(x) S(v,v')\,. \tag{3.10}$$

Proof. First, if $\|\sigma_\epsilon\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} \leq M$ and $\|k_\epsilon\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d\times\mathbf{R}^d)} \leq M$ for all $\epsilon > 0$, it is easily checked that f_ϵ extends to a function defined for negative times (still denoted by f_ϵ) satisfying the bound

$$\|f_\epsilon\|_{L^\infty([-T,T]\times\mathbf{R}^d\times\mathbf{R}^d)} \leq \|f^{in}\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} e^{M(1+V^d)T}\,. \tag{3.11}$$

Hence f_ϵ is relatively compact in $L^\infty([0,T]\times\mathbf{R}^d\times\mathbf{R}^d)$ weak-* for all $T > 0$. Let f be any limit point of this family as $\epsilon \to 0$; it is the limit of a weakly-* converging subsequence of the family (f_ϵ), denoted by $(f_{\epsilon'})$. It follows from the bound (3.11) and the equation (3.9) that, for all $T > 0$,

$$\|\partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon\|_{L^\infty([-T,T]\times\mathbf{R}^d\times\mathbf{R}^d)} \leq \|f^{in}\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} M(1+V^d) e^{M(1+V^d)T}\,. \tag{3.12}$$

By Proposition 2.3, for any $\chi \in C_0^\infty(\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d)$, one has

$$\int_{\mathbf{R}^d} f_\epsilon \chi dv \to \int_{\mathbf{R}^d} f\chi dv\,, \qquad \text{strongly in } L^1(\mathbf{R}\times\mathbf{R}^d)\,. \tag{3.13}$$

Applying then Proposition 2.2 shows that, as $\epsilon' \to 0$,

$$\sigma_{\epsilon'} f_{\epsilon'} \to \sigma f\,, \qquad k_{\epsilon'}(x,v,v') f_{\epsilon'}(t,x,v') \to k(x,v,v') f(t,x,v') \tag{3.14}$$

in $\mathcal{D}'(\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d)$ and $\mathcal{D}'(\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d\times\mathbf{R}^d)$. Using (3.4) shows that

$$\int_{\mathbf{R}^d} k_{\epsilon'}(x,v,v') f_{\epsilon'}(t,x,v') dv' \to \int_{\mathbf{R}^d} k(x,v,v') f(t,x,v') dv'$$

in $\mathcal{D}'(\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d)$ as $\epsilon' \to 0$. Hence f must satisfy (3.9). It also satisfies (3.8) since each term of the sequence $(f_{\epsilon'})$ does and because of the bound (3.12). Since the solution to the Cauchy problem (3.9)–(3.8) is unique, the proof is complete. ∎

One can observe that this result is much simpler than the one for diffusion equations: indeed, there is an explicit formula for the equivalent absorption and scattering cross sections, whereas there is no general formula for the H-limit (that is, equivalent diffusion matrix) of a sequence of oscillating matrices, apart from special cases, like that of dimension 1 or of laminated materials (see, for example, [28]).

THEOREM 3.2. *With the same assumptions and notations as in Theorem* 3.1, *but with* (3.4) *replaced by*

$$k_\epsilon(x,v,v') = 0 \quad \text{if } (v,v') \notin B(0,V)\times B(0,V) \tag{3.4'}$$

and with (3.6) *replaced by*
(3.6″)

$$\forall R>0\,,\quad \int_{|v'|\le R}\int_{|v|\le R}\int_{|x|\le R} |k_\epsilon(x,v,v') - k_\epsilon(x,v+w,v'+w')|dxdvdv' \to 0,$$

uniformly in ϵ *as* $|w|+|w'|\to 0$, $f_\epsilon \to f$ *in* $L^p_{loc}(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}^d)$ *strongly as* $\epsilon\to 0$ *for all* $1\le p<+\infty$.

Proof. Because of Theorem 3.1, we already know that $f_\epsilon \to f$ in $L^\infty(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}^d)$ weak-* as $\epsilon\to 0$. Clearly,
(3.15)

$$f_\epsilon(t,x,v)f_\epsilon(t,x,v') \to f(t,x,v)f(t,x,v') \text{ in } L^\infty(\mathbf{R}^+_t\times\mathbf{R}^d_x\times\mathbf{R}^d_v\times\mathbf{R}^d_{v'}) \text{ weak-*}\,.$$

Indeed, a small modification of (3.13) shows that, for any $\chi\in C_0^\infty(\mathbf{R}^+_t\times\mathbf{R}^d_x\times\mathbf{R}^d_v\times\mathbf{R}^d_{v'})$,

$$\int_{\mathbf{R}^d} f_\epsilon(t,x,v)\chi(t,x,v,v')dv \to \int_{\mathbf{R}^d} f(t,x,v)\chi(t,x,v,v')dv\,,$$

$$\text{strongly in } L^1(\mathbf{R}^+_t\times\mathbf{R}^d_x\times\mathbf{R}^d_{v'}) \tag{3.13'}$$

as $\epsilon\to 0$; hence by Proposition 2.2

$$\int_{\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}^d} f_\epsilon(t,x,v')\left(\int_{\mathbf{R}^d} f_\epsilon(t,x,v)\chi(t,x,v,v')dv\right)dtdxdv'$$

$$\to \int_{\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}^d} f(t,x,v')\left(\int_{\mathbf{R}^d} f(t,x,v)\chi(t,x,v,v')dv\right)dtdxdv'\,, \tag{3.16}$$

which establishes (3.15). Moreover, (3.7) shows that

$$\sup_\epsilon \|(\partial_t + (v+v')\cdot\nabla_x)(f_\epsilon(t,x,v)f_\epsilon(t,x,v'))\|_{L^\infty} < +\infty\,. \tag{3.17}$$

Now let $g(t,x,v)$ be the weak-* limit in $L^\infty(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}^d)$ of $(f_{\epsilon'}(t,x,v)^2)$, where ϵ' denotes a sequence of $]0,1[$ converging to 0. One has
(3.18)

$$\partial_t f^2_{\epsilon'} + v\cdot\nabla_x f^2_{\epsilon'} + 2\sigma_\epsilon(x,v)f^2_{\epsilon'} = 2\int_{\mathbf{R}^d} k(x,v,v')f_{\epsilon'}(t,x,v)f_{\epsilon'}(t,x,v')dv'\,,\quad x,v\in\mathbf{R}^d\,.$$

Applying Propositions 2.2 and 2.3 shows that the left side of (3.18) converges to

$$\partial_t g + v\cdot\nabla_x g + 2\sigma(x,v)g$$

in $\mathcal{D}(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}^d)$, while the right-hand side converges to

$$2\int_{\mathbf{R}^d} k(x,v,v')f(t,x,v)f(t,x,v')dv'\,.$$

Hence

$$\partial_t g + v \cdot \nabla_x g + 2\sigma(x,v)g = \partial_t f^2 + v \cdot \nabla_x f^2 + 2\sigma(x,v) f^2\,.$$

Since ${f^2}_{t=0} = g_{t=0}$, one necessarily has $f^2 = g$. By convexity of the map $f \mapsto f^2$, the announced strong convergence follows. (We wish to thank P. Gérard for suggesting this result and for a quick proof based on the notions developed in his paper [18]. We have given the proof above which is slightly more involved in order to keep the present paper self-contained). ∎

4. The case of periodic media. Although very general, the result obtained in the previous section can be rather deceiving. In particular, the formula for the equivalent cross sections is the same no matter what the geometry of the microstructure of the composite material may be. This indicates that the approximation consisting of replacing the solution of the Cauchy problem (3.7)–(3.8) by that of (3.9)–(3.8) can be very inaccurate, precisely because of its extreme generality. In the present section we shall work out an error estimate in the case of periodic composite materials.

To begin with, let us recall the basic framework of periodic homogenization. In the case of purely periodic composites, one assumes that the material consists of the periodic juxtaposition of identical microstructures containing the arrangement of the different components. The size of these microstructures is l. Hence, the cross sections of such composite media take the form

$$\sigma(x,v) = \Sigma\left(\frac{x}{l}, v\right)\,, \qquad k(x,v,v') = K\left(\frac{x}{l}, v, v'\right), \tag{4.1}$$

where Σ and K are functions defined on $\mathbf{T}^d \times \mathbf{R}^d$ and $\mathbf{T}^d \times \mathbf{R}^d \times \mathbf{R}^d$—or, if one prefers, functions of period 1 in their first variable. This is of course a very restrictive assumption on the mixture considered. It is possible, however, to gain some generality by distorting the arrangement of microstructures on a large scale, that is, on a scale of the same order as the macroscopic scale which is defined, for example, as the typical length of variation of the initial data. In this case, the cross sections of such composite media take the form

$$\sigma(x,v) = \Sigma\left(x, \frac{x}{l}, v\right)\,, \qquad k(x,v,v') = K\left(x, \frac{x}{l}, v, v'\right)\,, \tag{4.1$'$}$$

where Σ and K are functions defined on $\mathbf{R}^d \times \mathbf{T}^d \times \mathbf{R}^d$ and $\mathbf{R}^d \times \mathbf{T}^d \times \mathbf{R}^d \times \mathbf{R}^d$. Even with this generalization, the structure of the composite considered is very elementary and not much different from the purely periodic case.

We shall not dwell any longer on the mathematical formulation of periodic homogenization problems and refer instead to the treatise by Bensoussan–Lions–Papanicolaou [7]. Introducing the small parameter $\epsilon = l/L$, one sees that the problem under consideration is—at least if L remains of order 1, as we consistently assumed since the beginning of this work—

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \Sigma\left(x, \frac{x}{\epsilon}, v\right) f_\epsilon - \int_{\mathbf{R}^d} K\left(x, \frac{x}{\epsilon}, v, v'\right) f_\epsilon(t,x,v')dv' = 0\,, \qquad x, v \in \mathbf{R}^d\,, \tag{4.2}$$

$$f(0,x,v) = f^{in}(x,v)\,, \qquad (x,v) \in \mathbf{R}^d \times \mathbf{R}^d\,, \tag{4.3}$$

where Σ and K are functions defined on $\mathbf{R}^d \times \mathbf{T}^d \times \mathbf{R}^d$ and $\mathbf{R}^d \times \mathbf{T}^d \times \mathbf{R}^d \times \mathbf{R}^d$.

The traditional technique for such problems is to seek f_ϵ as a multiscale expansion of the form

$$f_\epsilon(t,x,v) \sim \sum_{m\geq 0} \epsilon^m f_m\left(t,x,\frac{x}{\epsilon},v\right) , \tag{4.4}$$

where f_m are functions defined on $\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{T}^d \times \mathbf{R}^d$.

Proceeding as in [7], we insert such an expansion in (4.2) and identify the coefficients of the successive powers of ϵ in the right-hand side of (4.2) to 0. This yields, denoting by y the periodic variable,

- Order ϵ^{-1}:

$$v \cdot \nabla_y f_0(t,x,y,v) = 0 , \tag{4.5}$$

- Order ϵ^0:

$$\partial_t f_0(t,x,y,v) + v\cdot\nabla_x f_0(t,x,y,v) + v\cdot\nabla_y f_1(t,x,y,v)$$

$$+\Sigma\left(x,y,v\right) f_0(t,x,y,v) - \int_{\mathbf{R}^d} K\left(x,y,v,v'\right) f_0(t,x,y,v')dv' = 0 . \tag{4.6}$$

The only solution to (4.5) is

$$f_0 \equiv f_0(t,x,v) . \tag{4.7}$$

This is most easily seen by writing (4.5) in the Fourier space: indeed (4.5) becomes

$$i2\pi v\cdot\xi \hat{f}_0(t,x,\xi,v) = 0, \tag{4.8}$$

where $\hat{f}_0$ designates the Fourier transform of f_0 in the periodic variable y only, keeping t, x, and v as fixed parameters. Now (4.8) must hold for almost every (t,x,v) and every $\xi \in \mathbf{Z}^d$. For almost every $v\in\mathbf{R}^d$ one has $v\cdot\xi\neq 0$ for all $\xi\in\mathbf{Z}^d\setminus\{0\}$. Hence

$$\hat{f}_0(t,x,\xi,v) = 0 , \qquad \text{almost everywhere (a.e.) in } (t,x,v) \quad \forall\xi\neq 0 . \tag{4.9}$$

By injectivity of the Fourier transform, the result follows immediately. At this point, we remind the reader that the multiscale expansion above is only a formal procedure; we shall see later that this expansion cannot in general be pursued after the first order terms in the case of the transport equation. Also, the "proof" that f_0 is independent of y requires that f_0 be at least, say, L^2 in (y,v).

It follows that (4.6) can be split as follows, after averaging (4.6) in the periodic variable y:

$$\partial_t f_0(t,x,v) + v\cdot\nabla_x f_0(t,x,v) + \langle\Sigma\left(x,\cdot,v\right)\rangle f_0(t,x,v) = \int_{\mathbf{R}^d} \langle K\left(x,\cdot,v,v'\right)\rangle f_0(t,x,v')dv' \tag{4.10}$$

and

$$v\cdot\nabla_y f_1(t,x,y,v) + \left(\Sigma\left(x,y,v\right) - \langle\Sigma\left(x,\cdot,v\right)\rangle\right) f_0(t,x,v)$$

$$-\int_{\mathbf{R}^d} \left(K\left(x,y,v,v'\right) - \langle K\left(x,\cdot,v,v'\right)\rangle\right) f_0(t,x,v')dv' = 0 . \tag{4.11}$$

At this point, (4.10) is precisely the homogenized equation obtained in the previous section. If one tries to imitate further the method of proof that works in the case of diffusion equations, one seeks to write

$$f_1(t,x,y,v) = -f_0(t,x,v)a(x,y,v) - \int_{\mathbf{R}^d} b(x,y,v,v')f_0(t,x,v')dv' \tag{4.12}$$

with a and b determined by the equations

$$v\cdot\nabla_y a(x,y,v) = \Sigma\,(x,y,v) - \langle\Sigma\,(x,\cdot,v)\rangle\,, \tag{4.13}$$

$$v\cdot\nabla_y b(x,y,v,v') = K\,(x,y,v,v') - \langle K\,(x,\cdot,v,v')\rangle\,. \tag{4.14}$$

This is where the case of linear transport equations differs from the case of diffusion equations in a most essential way. In the case of diffusion equations, one would have to solve the analogue of (4.13) with $v\cdot\nabla_y$ replaced by some uniformly elliptic self-adjoint second order differential operator on $\mathbf{T}^d$ in conservative form; since such operators are Fredholm in L^2 and have the space of constant (in y) functions as nullspace, it would suffice to check that the right-hand side is L^2 orthogonal to the space of constant (in y) functions, which is precisely the case since this right-hand side has average 0 in the variable y.

In the present case, (4.13) and (4.14) are not Fredholm, and writing these equations in the Fourier space shows exactly why. For example, (4.13) would be formally solved as

$$a(x,y,v) = \sum_{\xi\in\mathbf{Z}^d} \frac{\hat{\Sigma}(x,\xi,v)}{i2\pi\xi\cdot v} e^{i2\pi\xi\cdot v}, \tag{4.15}$$

and it is known that such series do not converge in any sense if one keeps a set of v's of full measure because of a small divisors problem: by the density of $\mathbf{Q}$ in $\mathbf{R}$, the denominators $i2\pi\xi\cdot v$ can become arbitrarily small.

However, we are not chiefly interested in writing the formal expansion (4.4) at any order; in fact, as long as one is interested in an error estimate, it suffices to stop at order 1 in this expansion. We shall see that the solution f_ϵ of the Cauchy problem (4.2)–(4.3) cannot have an asymptotic expansion of the form (4.4) at order higher than 1.

The idea consists of regularizing the "homological equations" (4.13)–(4.14) by adding some damping term where the damping rate will be taken as an appropriate function of the small parameter ϵ vanishing in the limit as $\epsilon\to 0$. A similar trick is known under the name of "limiting absorption principle" in the theory of scattering for, say, the wave equation. The main tool in this section is Lemma 4.1 below. Its statement requires introducing the following definition.

DEFINITION. *Let $R>0$; for all $A\in C_c^\infty(\mathbf{R}^m\times\mathbf{T}^d\times\mathbf{R}^d)$, consider*

$$|||A|||_R = \sup_z \left(\|A(z,\cdot,\cdot)\|_{L^\infty(\mathbf{T}^d\times\mathbf{R}^d)} + \sum_{\xi\in\mathbf{Z}^d} \sup_{|v|\le R} |\hat{A}(z,\xi,v)| \right)\,. \tag{4.16}$$

The completion of $C_c^\infty(\mathbf{R}^m\times\mathbf{T}^d\times\mathbf{R}^d)$ for this norm will further be denoted by $\mathcal{W}_R(\mathbf{R}^m\times\mathbf{T}^d\times\mathbf{R}^d)$. In other words, $\mathcal{W}_R(\mathbf{R}^m\times\mathbf{T}^d\times\mathbf{R}^d)$ is the class of tempered distributions on $\mathbf{R}^m\times\mathbf{T}^d\times\mathbf{R}^d$ with finite $|||\cdot|||_R$ norm.

For example, any function $A \equiv A(z,y) \in L^\infty(\mathbf{R}^m; C^{d+1}(\mathbf{T}^d))$ (viewed as constant in the velocity variable v) belongs to $\mathcal{W}_R(\mathbf{R}^m \times \mathbf{T}^d \times \mathbf{R}^d)$.

LEMMA 4.1. *Let $R > 0$. Let $A \equiv A(z,y,v)$ belong to $\mathcal{W}_R(\mathbf{R}^m \times \mathbf{T}^d \times \mathbf{R}^d)$. Assume that $\langle A(z,\cdot,v)\rangle = 0$ for all z and v and consider, for all $\lambda > 0$, the solution $\phi_\lambda(z,y,v)$ of*

$$\lambda\phi_\lambda + v \cdot \nabla_y \phi_\lambda = A\,. \tag{4.17}$$

Then, as $\lambda \to 0$, one has, for all $R > 0$,

$$\sup_{(z,y)\in\mathbf{R}^m\times\mathbf{T}^d} \|\phi_\lambda(z,y,\cdot)\|_{L^1(B(0,R))} = O\left(\log\left(\frac{1}{\lambda}\right)\right). \tag{4.18}$$

Proof. Write the solution of (4.17) as

$$\phi_\lambda(z,y,v) = \sum_{\xi\in\mathbf{Z}^d} \frac{\hat{A}(z,\xi,v)}{\lambda + i2\pi\xi\cdot v} e^{i2\pi\xi\cdot v}\,. \tag{4.19}$$

The bound (4.16) and the fact that A is of mean 0 show that

$$|\phi_\lambda(z,y,v)| \le \sum_{\xi\in\mathbf{Z}^d} \sup_v |\hat{A}(z,\xi,v)| \frac{1}{|\lambda + i2\pi\xi\cdot v|}\,.$$

Observe that

$$\int_{|v|\le R} \frac{dv}{|\lambda + i2\pi\xi\cdot v|} \le CR^{d-1}\int_{-R}^{R} \frac{dw}{|\lambda + i2\pi|\xi|w|}$$

$$= \frac{CR^{d-1}}{\pi|\xi|}\int_0^{2\pi|\xi|R/\lambda} \frac{du}{\sqrt{1+u^2}} = \frac{CR^{d-1}}{\pi|\xi|} O\left(\log\left(\frac{2\pi|\xi|R}{\lambda}\right)\right) \tag{4.20}$$

as $\lambda \to 0$. Hence

$$\int_{|v|\le R} |\phi_\lambda(z,y,v)|dv \le O\left(\log\left(\frac{1}{\lambda}\right)\right) \sum_{\xi\in\mathbf{Z}^d} \sup_{|v|\le R} |\hat{A}(z,\xi,v)|\,, \tag{4.21}$$

which, by (4.16), gives the announced result. ∎

THEOREM 4.2. *Let $R > 0$, and let Σ and K satisfy the following assumptions:*

$$K(x,y,v,v') = 0 \quad \text{if } |v| \text{ or } |v'| > R\,, \tag{4.22}$$

while

$$\Sigma \quad \text{and} \quad \nabla_x\Sigma \in \mathcal{W}_R(\mathbf{R}^d \times \mathbf{T}^d \times \mathbf{R}^d) \tag{4.23}$$

and

$$K \quad \text{and} \quad \nabla_x K \in \mathcal{W}_R(\mathbf{R}^d \times \mathbf{T}^d \times \mathbf{R}^d \times \mathbf{R}^d)\,. \tag{4.24}$$

Let f^{in} satisfy

$$f^{in}(x,v) = 0 \ \text{ if } |v| > R\,, \qquad \|f^{in}\|_{L^\infty} + \|\nabla_x f^{in}\|_{L^\infty} < +\infty, \tag{4.25}$$

and let f be the solution of the Cauchy problem

$$\partial_t f + v\cdot\nabla_x f + \langle\Sigma(x,\cdot,v)\rangle f - \int_{\mathbf{R}^d} \langle K(x,\cdot,v,v')\rangle f(t,x,v')dv' = 0\,, \quad x,v\in\mathbf{R}^d\,, \tag{4.26}$$

with initial condition (4.3)*. Then the solution of the Cauchy problem* (4.2)–(4.3) *satisfies, for all $T>0$ and all $0\le t\le T$,*

$$\|f_\epsilon(t,\cdot,\cdot) - f(t,\cdot,\cdot)\|_{L^1(\mathbf{R}^d\times\mathbf{R}^d)} = O(\epsilon|\log\epsilon|)\,. \tag{4.27}$$

Proof. To simplify, we shall give the proof in the case where the scattering cross section $k\equiv 0$. The generalization to the scattering cross sections considered in Theorem 4.2 is immediate. First, observe that condition (4.23) gives in particular

$$\|\Sigma\|_{L^\infty} + \|\nabla_x\Sigma\|_{L^\infty} \le M < +\infty\,. \tag{4.28}$$

Consider the solution $g_\epsilon \equiv g_\epsilon(t,x,v)$ of the Cauchy problem

$$\partial_t g_\epsilon(t,x,v) + v\cdot\nabla_x g_\epsilon(t,x,v) = -\Sigma\left(x,\frac{x}{\epsilon},v\right) g_\epsilon(t,x,v)\,,\ t\in\mathbf{R}\,,\ x,v,v'\in\mathbf{R}^d, \tag{4.29}$$

with initial data

$$g_\epsilon(0,x,v) = f^{in}(x,v)\,, \qquad x,v\in\mathbf{R}^d\,. \tag{4.30}$$

Consider now the solution g of the Cauchy problem

$$\partial_t g(t,x,v) + v\cdot\nabla_x g(t,x,v) = -\langle\Sigma(x,\cdot,v)\rangle g(t,x,v)\,,\ t\in\mathbf{R}\,,\ x,v\in\mathbf{R}^d, \tag{4.31}$$

with initial data (4.30). Now let

$$G_\epsilon(t,x,v) = g(t,x,v) + \epsilon g(t,x,v) a_\epsilon\left(x,\frac{x}{\epsilon},v\right), \tag{4.32}$$

where a_ϵ is the solution of

$$\epsilon a_\epsilon(x,y,v) + v\cdot\nabla_y a_\epsilon(x,y,v) = -\Sigma(x,y,v) + \langle\Sigma(x,\cdot,v)\rangle\,. \tag{4.33}$$

The function $R_\epsilon = g_\epsilon - G_\epsilon$ is a solution of

$$\begin{aligned} &\partial_t R_\epsilon(t,x,v) + v\cdot\nabla_x R_\epsilon(t,x,v) + \Sigma\left(x,\frac{x}{\epsilon},v\right) R_\epsilon(t,x,v) \\ &= -\epsilon\partial_t g(t,x,v) a_\epsilon\left(x,\frac{x}{\epsilon},v\right) - [v\cdot\nabla_x g(t,x,v)]\epsilon a_\epsilon\left(x,\frac{x}{\epsilon},v\right) \\ &\quad -g(t,x,v)\epsilon[v\cdot\nabla_x a_\epsilon]\left(x,\frac{x}{\epsilon},v\right)\,,\ t\in\mathbf{R}\,,\ x,v\in\mathbf{R}^d\,, \end{aligned} \tag{4.34}$$

with initial data

$$R_\epsilon(0,x,v) = -\epsilon f^{in}(x,v) a_\epsilon\left(x,\frac{x}{\epsilon},v\right)\,, \qquad x,v\in\mathbf{R}^d\,. \tag{4.35}$$

Call $S_\epsilon(t,x,v)$ the right-hand side of (4.34). First, the function g satisfies the bound

$$\|g\|_{L^\infty([-T,T]\times\mathbf{R}^d\times\mathbf{R}^d)} \le \|f\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} e^{MT}\,, \tag{4.36}$$

(4.37)
$$\|v\cdot\nabla_x g\|_{L^\infty([-T,T]\times\mathbf{R}^d\times\mathbf{R}^d)} \le (\|f\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} + \|\nabla_x f\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)})(1+M)e^{MT}\,.$$

Then, because of the support condition (4.25) on the initial data and the finite speed of propagation for the Cauchy problem (4.31)–(4.30), one has

$$g(t,x,v)=0 \quad \text{if } |x|>R(1+|t|)\,. \tag{4.38}$$

Formulas (4.34)–(4.36) and (4.37) give

$$|S_\epsilon(t,x,v)| \le C_T\left(\left|\epsilon a_\epsilon\left(x,\frac{x}{\epsilon},v\right)\right| + \left|\epsilon\nabla_x a_\epsilon\left(x,\frac{x}{\epsilon},v\right)\right|\right), \tag{4.39}$$

and (4.38) shows that

$$S_\epsilon(t,x,v)=0 \quad \text{if } |x|>R(1+|t|)\,. \tag{4.40}$$

Using (4.39)–(4.40) and Lemma 4.1 shows that

$$\|S_\epsilon(t,\cdot,\cdot)\|_{L^1(\mathbf{R}^d\times\mathbf{R}^d)} = O(\epsilon|\log\epsilon|)\,. \tag{4.41}$$

Likewise

$$\|R_\epsilon(0,\cdot,\cdot)\|_{L^1(\mathbf{R}^d\times\mathbf{R}^d)} = O(\epsilon|\log\epsilon|)\,. \tag{4.42}$$

Applying the Duhamel formula for the Cauchy problem (4.34)–(4.35) shows that

$$\begin{aligned}\|R_\epsilon(t,\cdot,\cdot)\|_{L^1(\mathbf{R}^d\times\mathbf{R}^d)} &\le e^{Mt}\|R_\epsilon(0,\cdot,\cdot)\|_{L^1(\mathbf{R}^d\times\mathbf{R}^d)}\\ &+\frac{e^{Mt}-1}{M}\sup_{0\le t\le T}\|S_\epsilon(t,\cdot,\cdot)\|_{L^1(\mathbf{R}^d\times\mathbf{R}^d)} = O(\epsilon|\log\epsilon|)\,.\end{aligned} \tag{4.43}$$

Again applying Lemma 4.1 shows that, for all $T>0$ and $0\le t\le T$, one has

$$\|g_\epsilon(t,\cdot,\cdot)-g(t,\cdot,\cdot)\|_{L^1(\mathbf{R}^d\times\mathbf{R}^d)} = O(\epsilon|\log\epsilon|) \tag{4.44}$$

as announced. ∎

Remark. Estimate (4.27) is an upper bound for the error estimate. It might be that, for particular classes of absorption and scattering cross sections Σ and K, equations (4.13)–(4.14) indeed have solutions a and b in some appropriate L^p space. That would allow expanding f_ϵ as in (4.4) up to order 1 in ϵ. As a matter of fact, the unbounded skew adjoint operator $v\cdot\nabla_y$ acting on $L^2(\mathbf{T}^d\times\mathbf{R}^d)$ has nullspace reduced to L^2 functions of v alone. Its range is therefore everywhere dense in, but not equal to, the closed subspace E of functions $S\equiv S(y,v)$ in $L^2(\mathbf{T}^d\times\mathbf{R}^d)$ such that $\int_{\mathbf{T}^d} S(y,v)dy=0$ for a.e. $v\in\mathbf{R}^d$. To disprove equality, it suffices to apply the open mapping theorem to the operator $v\cdot\nabla_y(-\Delta_y)^{1/2}$ on E: indeed, the sequence of functions $g_n(y,v)=\sqrt{n}\mathbf{1}_{|v|\le1}\mathbf{1}_{n|v_1|\le1}\cos y_1$ satisfies

$$v\cdot\nabla_y(-\Delta_y)^{1/2}g_n\to0 \text{ while } g_n \text{ does not converge to 0 in } L^2(\mathbf{T}^d\times\mathbf{R}^d)$$

as $n\to+\infty$. Thus the operator $v\cdot\nabla_y(-\Delta_y)^{1/2}$ cannot be onto on E. The proof of Theorem 4.2 suggests that, unless the functions

$$(y,v)\mapsto\Sigma(x,y,v)-\langle\Sigma(x,\cdot,v)\rangle\,,\qquad (y,v)\mapsto K(x,y,v,v')-\langle K(x,\cdot,v,v')\rangle$$

belong to the range of $v \cdot \nabla_y$, the solution of (4.2)–(4.3) might admit no asymptotic expansion of the form (4.4).

Remark. The L^1 norm is the one for which the best error estimate is obtained (in terms of the order in ϵ). Under the same assumptions, the same strategy as above with obvious modifications would give

$$\|g_\epsilon(t,\cdot,\cdot) - g(t,\cdot,\cdot)\|_{L^2(\mathbf{R}^d\times\mathbf{R}^d)} = O(\epsilon^{1/2})\,.$$

5. The case of random media. In the last section of this paper, we study the homogenization of transport equations in certain kinds of random media. The random media considered in this paper are somewhat different from the ones considered in [25], [26]. These references defined the composite materials they considered by the distribution of inhomogeneities along straight lines. While this is obviously well adapted to the transport of particles, it may seem a little artificial. We have chosen to consider here the type of random media for which the homogenization problem is already known in the case of diffusion equations (see [31]).

Presentation of the random media considered. Consider the model due to Papanicolaou–Varadhan [31]. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\sigma(y,\omega))_{y\in\mathbf{R}^d}$ be a stationary family of nonnegative (real-valued) random variables on Ω. In other words,

(H1) for each $p \in \mathbf{N}^*$, each $h \in \mathbf{R}^d$ and each $(y_1,\dots,y_p) \in (\mathbf{R}^d)^p$, the joint distribution of $\sigma(y_1,\omega),\dots,\sigma(y_p,\omega)$ is equal to that of $\sigma(y_1+h,\omega),\dots,\sigma(y_p+h,\omega)$.

Denoting by $\mathbf{E}$ the expectation under P, let $\mathcal{H} = L^2(\Omega,\mathcal{F},P)$ equipped with the scalar product

$$\langle f, g\rangle_{\mathcal{H}} = \mathbf{E}(fg)\,.$$

We shall make the assumption that the probability space $(\Omega,\mathcal{F},P)$ is equipped with a group of one–one transformations parametrized by $\mathbf{R}^d$ leaving the probability P invariant. In other words, we assume the existence, for all $x \in \mathbf{R}^d$, of a one–one transformation $\tau_x : \Omega \to \Omega$ such that

$$P({\tau_x}^{-1}G) = P(G) \quad \forall G \in \mathcal{F}\,,\ \forall x \in \mathbf{R}^d\,, \tag{5.1}$$

with the notation

$$ {\tau_x}^{-1}G = \{\omega \in \Omega \,/\, \tau_x\omega \in G\} \tag{5.2}$$

and such that

$$\tau_x \circ \tau_y = \tau_{x+y} \quad \forall x,y \in \mathbf{R}^d\,. \tag{5.3}$$

To the group of measure preserving one–one transformations (τ_x) is associated the group of unitary transformations $T_x : \mathcal{H} \to \mathcal{H}$ defined by

$$(T_x\tilde f)(\omega) = \tilde f(\tau_{-x}\omega) \quad x \in \mathbf{R}^d, \quad \tilde f \in \mathcal{H}. \tag{5.4}$$

The infinitesimal generators of this group of unitary transformations are defined as usual by

$$\partial_{x_i}(T_x\tilde f)_{|x=0} = D_i\tilde f\,, \qquad (i = 1,\dots,d), \tag{5.5}$$

and the domains $\mathcal{D}_i$ of the unbounded operator D_i on $\mathcal{H}$ are defined to be the set of $\tilde f$'s such that the right-hand side of the equality above (i.e., the derivative at $x = 0$)

is defined. These domains are dense subspaces of $\mathcal{H}$, whose intersection is denoted by $\mathcal{H}^1$ and equipped with the structure of Hilbert space defined by

$$\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^1} = \langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}} + \sum_{i=1}^{d} \langle D_i \tilde{f}, D_i \tilde{g} \rangle_{\mathcal{H}} . \tag{5.6}$$

We shall denote by U the spectral decomposition of the identity associated to the unitary group T: U is a measure on $\mathbf{R}^d$ with values in the set of orthogonal projections of $\mathcal{H}$ such that

$$T_x = \int_{\mathbf{R}^d} e^{i\lambda \cdot x} U(d\lambda) . \tag{5.7}$$

The example of a Poisson distribution of inhomogeneities. This particular case is usually considered as important for physical applications; see [30]. It is formulated as follows: let $\Omega = \{\text{countable subsets of } \mathbf{R}^d\}$. The Poisson probability P on Ω is defined by the relation

$$\forall A \subset \mathbf{R}^d , \qquad P(\{\omega \in \Omega \,|\, \sharp(\omega \cap A) = n\}) = e^{-\lambda |A|} \frac{\lambda^n |A|^n}{n!} .$$

The action of $\mathbf{R}^d$ on Ω is defined by

$$\tau_x(\omega) = x + \omega .$$

Then let $\tilde{\sigma} : \Omega \to \mathbf{R}$ be such that

$$\tilde{\sigma}(\omega) = \sigma_1 \text{ if } \omega \cap B(0, \delta) \neq \emptyset ,$$

$$\tilde{\sigma}(\omega) = \sigma_2 \text{ if } \omega \cap B(0, \delta) = \emptyset .$$

One can view $\sigma(x, \omega) = \tilde{\sigma}(\tau_x^{-1}(\omega))$ as a cross section for a composite material consisting of spherical inclusions distributed according to the Poisson law (with possible overlapping) in some background material.

We shall establish the homogenization property for random media such that the absorption cross section σ is a random function satisfying (H1); such random media are called homogeneous. However, the homogenization procedure will not apply to all such media; it will be necessary to introduce a further assumption, namely that the group of transformations (τ_x) be ergodic:

(H2) for any $x \neq 0$, the only $\mathcal{F}$-measurable subsets of Ω invariant under τ_x, i.e., such that $G \subset {\tau_x}^{-1} G$, have probability 0 or 1.

As usual, this can be rephrased in terms of the unitary group (T_x) acting on $\mathcal{H}$, as follows:

(H2)$'$ for any $x \neq 0$, the only functions in $\mathcal{H}$ invariant under T_x are the (almost surely (a.s.)) constant functions.

We recall then the following two forms of the ergodic theorem:

- the von Neumann form:

$$\forall \tilde{f} \in \mathcal{H} , \qquad \frac{1}{(2n)^N} \int_{[-n,n]^d} (T_x \tilde{f})(\omega) dx \to \mathbf{E}(\tilde{f}) \text{ in } \mathcal{H} \text{ as } n \to \infty ; \tag{5.8}$$

- the Birkhoff form:

$$\forall \tilde{f} \in L^1(\Omega, \mathcal{F}, P) , \qquad \text{the convergence above holds } P \text{ a.s. on } \Omega. \tag{5.9}$$

Once there is given a probability space $(\Omega, \mathcal{F}, P)$ and a group of one–one transformations $(\tau_x)_{x\in\mathbf{R}^d}$ verifying (5.1)–(5.3) and (H2), the simplest example of random function σ verifying (H1) is given by

$$\sigma(y,\omega) = \tilde{\sigma}(\tau_{-y}\omega), \tag{5.10}$$

where $\tilde{\sigma}$ is a bounded function on Ω (and therefore defines an element of $\mathcal{H}$).

The strong convergence result. In the following, we shall consider a probability space $(\Omega, \mathcal{F}, P)$ and a group of one–one transformations $(\tau_x)_{x\in\mathbf{R}^d}$ verifying (5.1)–(5.3) and (H2), and a linear transport equation of the form

$$\begin{aligned}\partial_t f_\epsilon(t,x,v,\omega) + v\cdot\nabla_x f_\epsilon(t,x,v,\omega) + \sigma\left(\frac{x}{\epsilon},\omega\right) f_\epsilon(t,x,v,\omega)\\ = \sigma\left(\frac{x}{\epsilon},\omega\right)\int_{\mathbf{R}^d} k(v,v')f_\epsilon(t,x,v',\omega)d\mu(v'),\\ t\in\mathbf{R}^+\,,\ x\in\mathbf{R}^d\,,\ v\in\mathbf{R}^d\,,\ \omega\in\Omega,\end{aligned} \tag{5.11}$$

where σ is a random function satisfying (H1) and (5.10), k is a compactly supported bounded function on $\mathbf{R}^d\times\mathbf{R}^d$ and μ is a positive bounded measure on $\mathbf{R}^d$. We fix the initial condition to be

$$f_\epsilon(0,x,v,\omega) = f^{in}(x,v) \quad x\in\mathbf{R}^d\,,\ v\in\mathbf{R}^d\,,\ \omega\in\Omega\,. \tag{5.12}$$

THEOREM 5.1. *Let the probability space* $(\Omega, \mathcal{F}, P)$ *and the group of one–one transformations* $(\tau_x)_{x\in\mathbf{R}^d}$ *verify* (5.1)–(5.3) *and* (H2), *and let the random function* σ *be given by* (5.10). *Assume that the measure* μ *is bounded and satisfies*

$$\mu(\{v\in\mathbf{R}^d\,|\,|v\cdot\xi|\le\alpha|\xi|\}\le C\alpha^\gamma \tag{\textbf{H3}}$$

for some values of $0<\gamma<1$ *and* $C>0$, *uniformly in* $\xi\in\mathbf{R}^d$ *and* $\alpha\ge 0$. *Then, for all* $f^{in}\in L^\infty(\mathbf{R}^d_v;W^{1,\infty}(\mathbf{R}^d_x))$ *and all* $T>0$, f_ϵ *converges in* $L^\infty([0,T]\times\mathbf{R}^d_x;L^2(\mathbf{R}^d_v,\mathcal{H}))$ *to the solution* f *of the Cauchy problem*

$$\begin{aligned}\partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) + \overline{\sigma}\left(f(t,x,v) - \int_{\mathbf{R}^d} k(v,v')f(t,x,v')d\mu(v')\right) = 0\,,\\ t\in\mathbf{R}_+\,,\ x,v\in\mathbf{R}^d\end{aligned} \tag{5.13}$$

with initial condition (5.12) *and the notation*

$$\overline{\sigma} = \mathbf{E}(\sigma)\,.$$

As in the case of periodic inhomogeneities, the proof will rest on the following result of the "damped homological equation."

LEMMA 5.2. *Let* $\tilde{\Sigma}(\omega) = \tilde{\sigma}(\omega) - \overline{\sigma}$, *where* $\tilde{\sigma}$ *is a uniformly bounded function on* Ω. *For all* $\lambda>0$, *let* $\tilde{\chi}_\lambda(v,\cdot)\in\mathcal{H}$ *be the solution of*

$$\sum_{i=1}^d v_i\cdot D_i\tilde{\chi}_\lambda(v,\omega) + \tilde{\Sigma}(\omega) = \lambda\tilde{\chi}_\lambda(v,\omega) \quad v\in\mathbf{R}^d, \quad \omega\in\Omega\,. \tag{5.14}$$

Under assumptions (H2)–(H3), *as* $\lambda \to 0^+$, *one has*

$$\lambda \tilde{\chi}_\lambda \to 0 \quad \text{in } L^2(\mathbf{R}^d, \mathcal{H}) \tag{5.15}$$

and

$$\sup_{y \in \mathbf{R}^d} \lambda \|\chi_\lambda(y, \cdot, \cdot)\|_{L^2(\mathbf{R}^d, \mathcal{H})} \to 0\,, \tag{5.16}$$

with the notation

$$\chi(y, \omega) = \tilde{\chi}(\tau_{-y}\omega)\,. \tag{5.17}$$

Proof. One has

$$\tilde{\chi}_\lambda(v, \omega) = \int_0^{+\infty} e^{-\lambda t}(T_{tv}\tilde{\Sigma})(\omega)dt, \tag{5.18}$$

which, by using the spectral decomposition (5.7) of T_x, can be recast as

$$\tilde{\chi}_\lambda(v, \omega) = \int_{\mathbf{R}^d} \frac{1}{\lambda + iv\cdot\beta} U(d\beta)\tilde{\Sigma}(\omega)\,. \tag{5.19}$$

This immediately gives the following formula:

$$\begin{aligned} I_\lambda = \int_{\mathbf{R}^d} \lambda^2 \|\tilde{\chi}_\lambda(v, \cdot)\|^2_{\mathcal{H}} d\mu(v) &= \int_{\mathbf{R}^d} d\mu(v) \int_{\mathbf{R}^d} \frac{\lambda^2}{\lambda^2 + |v\cdot\beta|^2} \langle U(d\beta)\tilde{\Sigma}, \tilde{\Sigma}\rangle_{\mathcal{H}} \\ &= \int_{\mathbf{R}^d} \langle U(d\beta)\tilde{\Sigma}, \tilde{\Sigma}\rangle_{\mathcal{H}} \int_{|v\cdot\beta| > \alpha|\beta|} \frac{\lambda^2}{\lambda^2 + |v\cdot\beta|^2} d\mu(v) \\ &\quad + \int_{\mathbf{R}^d} \langle U(d\beta)\tilde{\Sigma}, \tilde{\Sigma}\rangle_{\mathcal{H}} \int_{|v\cdot\beta| \le \alpha|\beta|} \frac{\lambda^2}{\lambda^2 + |v\cdot\beta|^2} d\mu(v) \end{aligned}$$

for all $\alpha > 0$. Hence

$$I_\lambda \le \int_{\mathbf{R}^d} d\mu(v) \int_{\mathbf{R}^d} \frac{\lambda^2}{\lambda^2 + \alpha^2|\beta|^2} \langle U(d\beta)\tilde{\Sigma}, \tilde{\Sigma}\rangle_{\mathcal{H}} + C\alpha^\gamma \|\tilde{\Sigma}\|^2_{\mathcal{H}}\,, \tag{5.20}$$

thanks to (H3). Now let $\alpha > 0$ be fixed; applying the dominated convergence theorem to the first term in the right-hand side of the inequality (5.20) above shows that, as $\lambda \to 0$,

$$\int_{\mathbf{R}^d} d\mu(v) \int_{\mathbf{R}^d} \frac{\lambda^2}{\lambda^2 + \alpha^2|\beta|^2} \langle U(d\beta)\tilde{\Sigma}, \tilde{\Sigma}\rangle_{\mathcal{H}} \to \mu(\mathbf{R}^d)\langle U(\{0\})\tilde{\Sigma}, \tilde{\Sigma}\rangle_{\mathcal{H}} = 0 \tag{5.21}$$

by assumption (H2) since $U(\{0\})$ is the orthogonal projection on the functions left invariant by T_x for $x \neq 0$, that is, on the space of a.s. constant functions, and $\tilde{\Sigma}$ is of mean zero. Hence

$$\limsup_{\lambda \to 0} I_\lambda \le C\alpha^\gamma \|\tilde{\Sigma}\|^2_{\mathcal{H}}, \tag{5.22}$$

and since this inequality holds for all $\alpha > 0$, (5.15) follows. The convergence (5.16) is a direct consequence of (5.15) with the definition (5.17). $\square$

Proof of Theorem 5.1. This proof follows closely that of Theorem 4.2. Let

$$R_\epsilon(t,x,v,\omega) = f_\epsilon(t,x,v,\omega) - f(t,x,v)$$

$$-\epsilon\left(f(t,x,v) - \int_{\mathbf{R}^d} k(v,v')f(t,x,v')d\mu(v')\right)\chi_\epsilon\left(\frac{x}{\epsilon},v,\omega\right). \tag{5.23}$$

We compute, for all $t \in \mathbf{R}^+,\ x \in \mathbf{R}^d,\ v \in \mathbf{R}^d,\ \omega \in \Omega$, the expression

$$S_\epsilon(t,x,v,\omega) = \partial_t R_\epsilon + v\cdot\nabla_x R_\epsilon$$

$$+\sigma\left(\frac{x}{\epsilon},\omega\right)\left(R_\epsilon(t,x,v,\omega) - \int_{\mathbf{R}^d} k(v,v')R_\epsilon(t,x,v',\omega)d\mu(v')\right) \tag{5.24}$$

with

$$S_\epsilon(t,x,v,\omega) = -\epsilon\chi_\epsilon\left(\frac{x}{\epsilon},v,\omega\right)[1+(\partial_t + v\cdot\partial_x)]Z(t,x,v)$$

$$-\epsilon\sigma\left(\frac{x}{\epsilon},\omega\right)\chi_\epsilon\left(\frac{x}{\epsilon},v,\omega\right)Z(t,x,v)$$

$$+\epsilon\sigma\left(\frac{x}{\epsilon},\omega\right)\int_{\mathbf{R}^d} k(v,v')\chi_\epsilon\left(\frac{x}{\epsilon},v',\omega\right)Z(t,x,v')d\mu(v') \tag{5.25}$$

with

$$Z(t,x,v) = f(t,x,v) - \int_{\mathbf{R}^d} k(v,v')f(t,x,v')d\mu(v').$$

The corresponding initial condition for the remainder term is

$$R_\epsilon(0,x,v,\omega) = -\epsilon\chi_\epsilon\left(\frac{x}{\epsilon},v,\omega\right)\left(f(t,x,v) - \int_{\mathbf{R}^d} k(v,v')f(t,x,v')d\mu(v')\right). \tag{5.26}$$

Applying the Duhamel formula to the Cauchy problem (5.24)–(5.26) and the maximum principle to the limiting equation (5.13) with the initial condition (5.12) gives

$$\begin{aligned}\|R_\epsilon(t,\cdot,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d_x;L^2(\mathbf{R}^d_v;\mathcal{H}))} \le & 2\|f^{in}\|_{L^\infty(\mathbf{R}^d_v;W^{1,\infty}(R^d_x))}\|\epsilon\chi_\epsilon(\frac{\cdot}{\epsilon},\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d_x;L^2(\mathbf{R}^d_v;\mathcal{H}))}\\ & + t\sup_{0\le s\le t}\|S_\epsilon(s,\cdot,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d_x;L^2(\mathbf{R}^d_v;\mathcal{H}))},\end{aligned}$$

and it follows immediately from Lemma 5.2 that, for all $T>0$, as $\epsilon \to 0$,

$$\|R_\epsilon(t,\cdot,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d_x;L^2(\mathbf{R}^d_v;\mathcal{H}))} \to 0.$$

Again applying Lemma 5.2 to the last term in the right-hand side of (5.23), one gets

$$\|f_\epsilon(t,\cdot,\cdot,\cdot) - f(t,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d_x;L^2(\mathbf{R}^d_v;\mathcal{H}))} \to 0$$

as $\epsilon \to 0$. $\square$

Error estimate. In order to get a sharper result, we shall assume that the action of $\mathbf{R}^d$ on Ω is mixing and not only ergodic. This assumption bears on the decay of the self-correlation function of the absorption cross section Σ along trajectories of the free transport equation.

Let $C^v_{\tilde{\Sigma}} = \langle T_{tv}\tilde{\Sigma}, \tilde{\Sigma}\rangle_{\mathcal{H}}$. The mixing assumption is

$$\textbf{(H4)} \qquad \int_{\mathbf{R}^d} C^v_{\tilde{\Sigma}}(t)d\mu(v) = O(t^{-\alpha})$$

as $t \to \infty$ for some $\alpha > 1$.

THEOREM 5.3. *Let $f_\epsilon \equiv f_\epsilon(t,x,v,\omega)$ be the solution of the Cauchy problem* (5.11)–(5.12). *If one keeps the same assumptions as in Theorem* 5.3 *except for* (H2), *which is replaced by the stronger assumption* (H4), *then, for all $T > 0$, there exists $C_T > 0$ such that*

$$\|f_\epsilon(t,\cdot,\cdot,\cdot) - f(t,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d_x; L^2(\mathbf{R}^d_v;\mathcal{H}))} \le C_T\sqrt{\epsilon},$$

uniformly in $t \in [0,T]$, where f is the solution of the Cauchy problem (5.13)–(5.12).

Proof. The proof follows that of Theorem 5.1. The only difference consists of the estimate of χ_λ. To do this, write

$$\begin{aligned}
\sqrt{\lambda}\tilde{\chi}_\lambda(v,\omega) &= \int_0^\infty \sqrt{\lambda}e^{-\lambda t}(T_{tv}\tilde{\Sigma})(\omega)dt \\
&= \left[\sqrt{\lambda}e^{-\lambda t}\int_0^t (T_{sv}\tilde{\Sigma})(\omega)ds\right]_0^\infty + \int_0^\infty \lambda^{\frac{3}{2}}e^{-\lambda t}\int_0^t (T_{sv}\tilde{\Sigma})(\omega)dsdt.
\end{aligned}$$

Since $(T_{sv}\tilde{\Sigma})(\omega) = \tilde{\Sigma}(\tau_{-sv}\omega)$ is a bounded function of s,

$$\begin{aligned}
\sqrt{\lambda}\tilde{\chi}_\lambda(v,\omega) &= \int_0^\infty \lambda^{\frac{3}{2}}e^{-\lambda t}\int_0^t (T_{sv}\tilde{\Sigma})(\omega)dsdt \\
&= \int_0^\infty \sqrt{\lambda t}e^{-\lambda t}\left(\frac{1}{\sqrt{t}}\int_0^t (T_{sv}\tilde{\Sigma})(\omega)ds\right)\lambda dt \\
&= \int_0^\infty \sqrt{\theta}e^{-\theta}\sqrt{\frac{\lambda}{\theta}}\int_0^{\frac{\theta}{\lambda}} (T_{sv}\tilde{\Sigma})(\omega)dsd\theta.
\end{aligned}$$

The Cauchy–Schwarz inequality gives

$$\lambda\int_{\mathbf{R}^d}\|\tilde{\chi}_\lambda(v,\cdot)\|^2_{\mathcal{H}}d\mu(v)$$

$$\le \Gamma\left(\frac{3}{2}\right)\int_V dv\int_0^\infty \sqrt{\theta}e^{-\theta}\mathbf{E}\left|\sqrt{\frac{\lambda}{\theta}}\int_0^{\frac{\theta}{\lambda}}(T_{sv}\tilde{\Sigma})(\omega)ds\right|^2 d\theta\,.$$

It is classical to express $\mathbf{E}\left|\frac{1}{\sqrt{t}}\int_0^t(T_{sv}\tilde{\Sigma})ds\right|^2$ in terms of $C^v_{\tilde{\Sigma}}(s)$:

$$\begin{aligned}\mathbf{E}\left|\frac{1}{\sqrt{t}}\int_0^t(T_{sv}\tilde{\Sigma})ds\right|^2 &= \frac{1}{t}\mathbf{E}\left(\int_0^t\int_0^t(T_{sv}\tilde{\Sigma})(T_{\theta v}\tilde{\Sigma})d\theta ds\right)\\ &= \frac{2}{t}\mathbf{E}\left(\int_0^t\int_0^s(T_{sv}\tilde{\Sigma})(T_{\theta v}\tilde{\Sigma})d\theta ds\right)\\ &= \frac{2}{t}\int_0^t\int_0^s\mathbf{E}\left((T_{sv}\tilde{\Sigma})(T_{\theta v}\tilde{\Sigma})\right)d\theta ds\\ &= \frac{2}{t}\int_0^t\int_0^s C^v_{\tilde{\Sigma}}(s-\theta)d\theta ds\end{aligned}$$

since P is invariant under $\tau_{\theta v}$. Therefore,

$$\begin{aligned}\mathbf{E}\left|\frac{1}{\sqrt{t}}\int_0^t(T_{sv}\tilde{\Sigma})ds\right|^2 &= \frac{2}{t}\int_0^t C^v_{\tilde{\Sigma}}(\theta)\left(\int_\theta^t ds\right)d\theta\\ &= 2\int_0^t C^v_{\tilde{\Sigma}}(\theta)d\theta - \frac{2}{t}\int_0^t\theta C^v_{\tilde{\Sigma}}(\theta)d\theta.\end{aligned}$$

In the last inequality, the mixing assumption (H4) ensures that

$$t\mapsto\int_{\mathbf{R}^d}\mathbf{E}\left|\frac{1}{\sqrt{t}}\int_0^t(T_{sv}\tilde{\Sigma})ds\right|^2 d\mu(v)\in L^\infty([0,+\infty)).$$

Hence there exists a constant $C>0$ such that

$$\int_{\mathbf{R}^d}d\mu(v)\lambda\mathbf{E}|\tilde{\chi}_\lambda(v,\omega)|^2\le C\,.$$

In particular,

$$\sup_{y\in\mathbf{R}^D}\|\sqrt{\lambda}\chi_\lambda(y,.)\|_{L^2(\mathbf{R}^d_v,\mathcal{H})}$$

is bounded uniformly in λ. Theorem 5.3 follows from here in the same way Theorem 5.1 follows from Lemma 5.2. $\square$

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