Vers une meilleure détection des troubles de la vision

ou: comment les mathématiques peuvent venir en aide à la médecine

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Partie 1: le problème médical

Partie 2: traduction mathématique

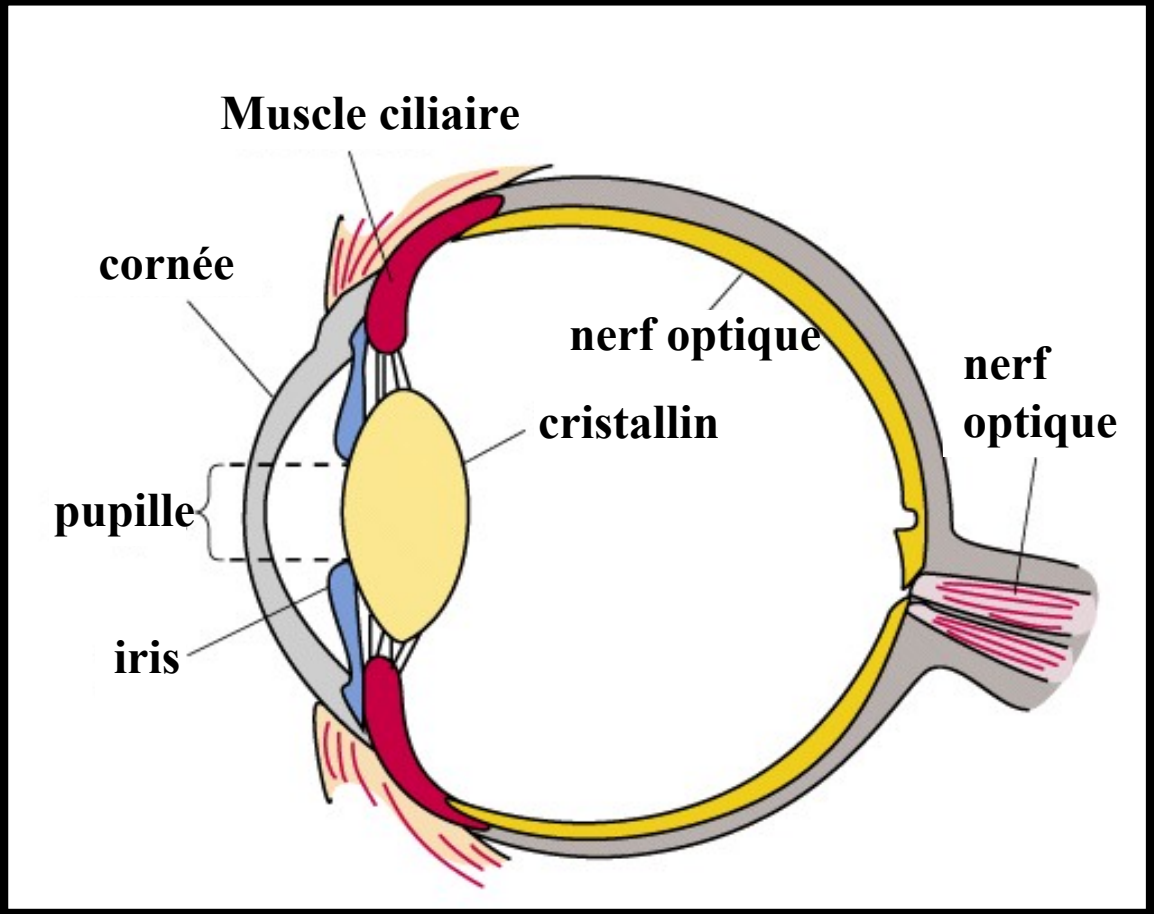
Partie 3: les résultats cliniques

Conclusion et problèmes en cours

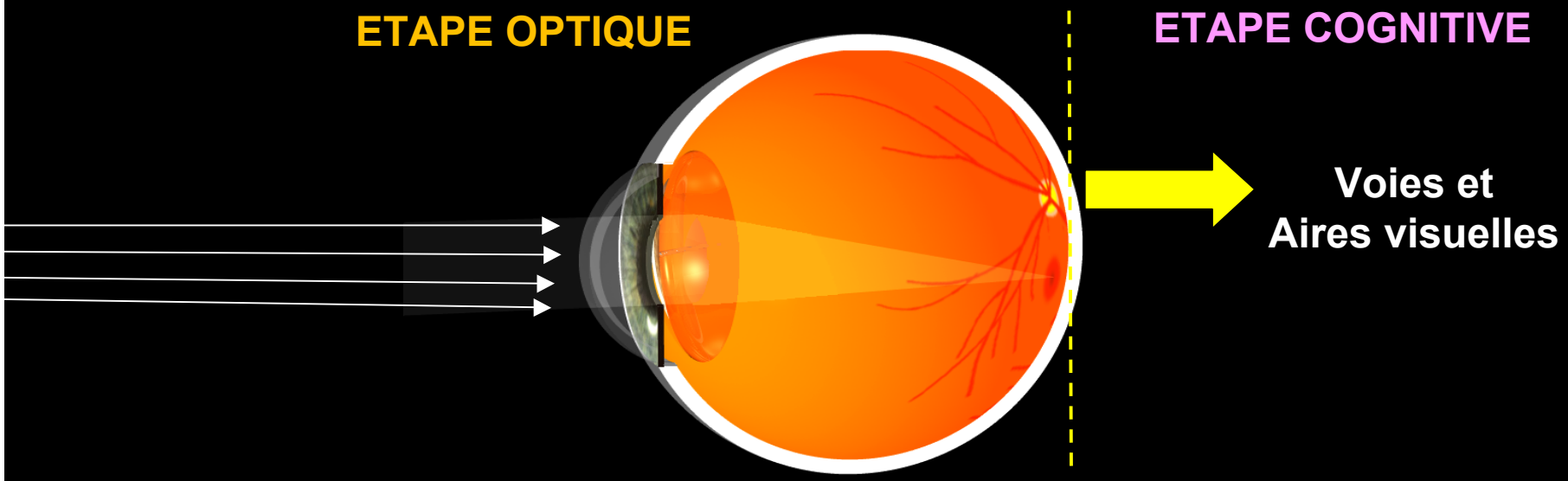
Partie 1: le problème médical

LA VISION

OEIL HUMAIN: ORGANE SENSORIEL - CAPTEUR POUR LA VISION



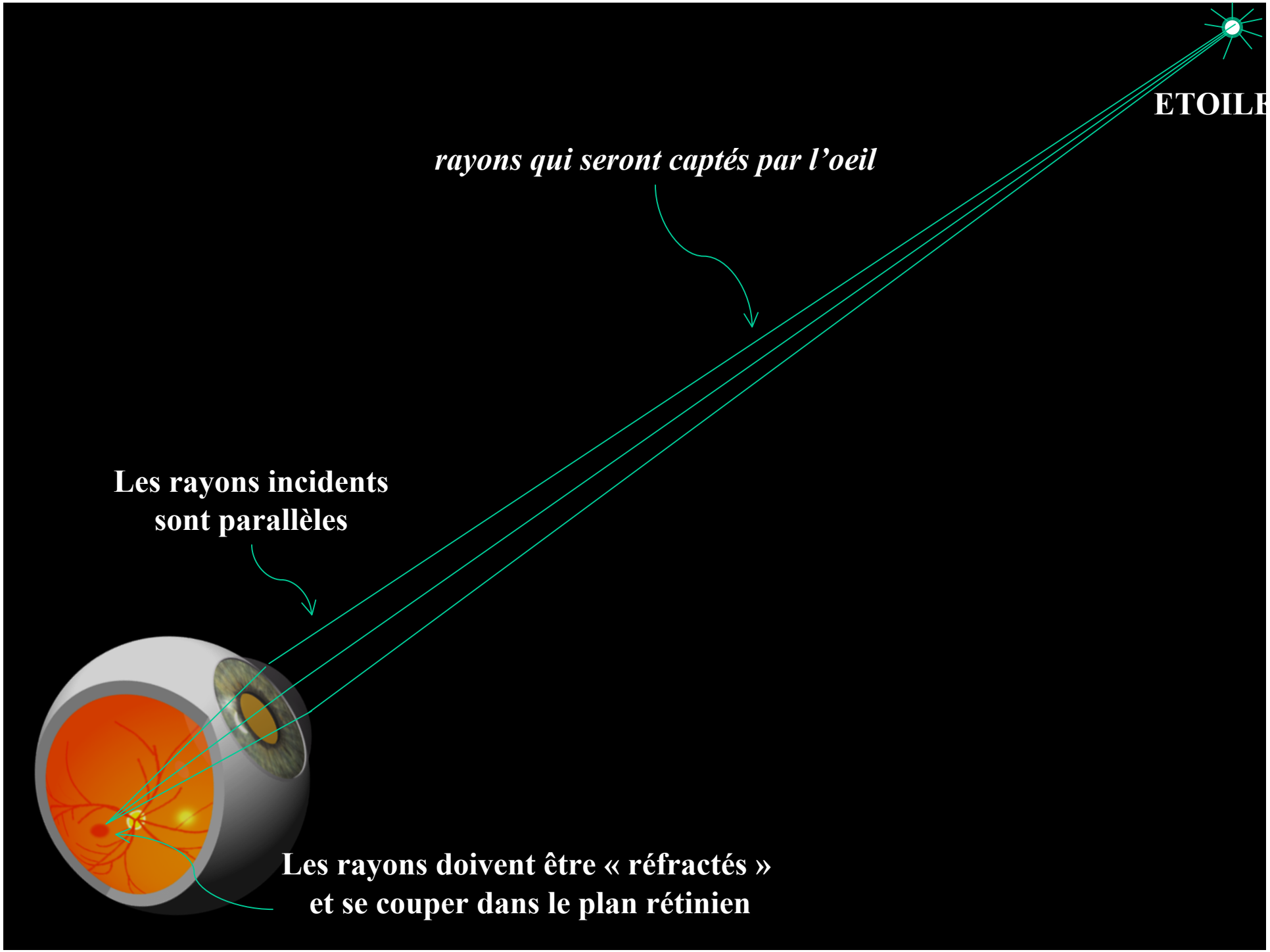
LA VISION



VISION : un processus complexe que l'on peut décomposer en deux étapes principales :

- 1) acquisition de l'information optique
(ETAPE OPTIQUE : transport des photons de la cornée vers les photorécepteurs)

- 2) transmission et interprétation de l'information visuelle
(ETAPE NEURO-COGNITIVE : l'énergie photonique est convertie en courant axonal)


ETOILE
rayons qui seront captés par l'oeil
Les rayons incidents
sont parallèles
Les rayons doivent être « réfractés »
et se couper dans le plan rétinien

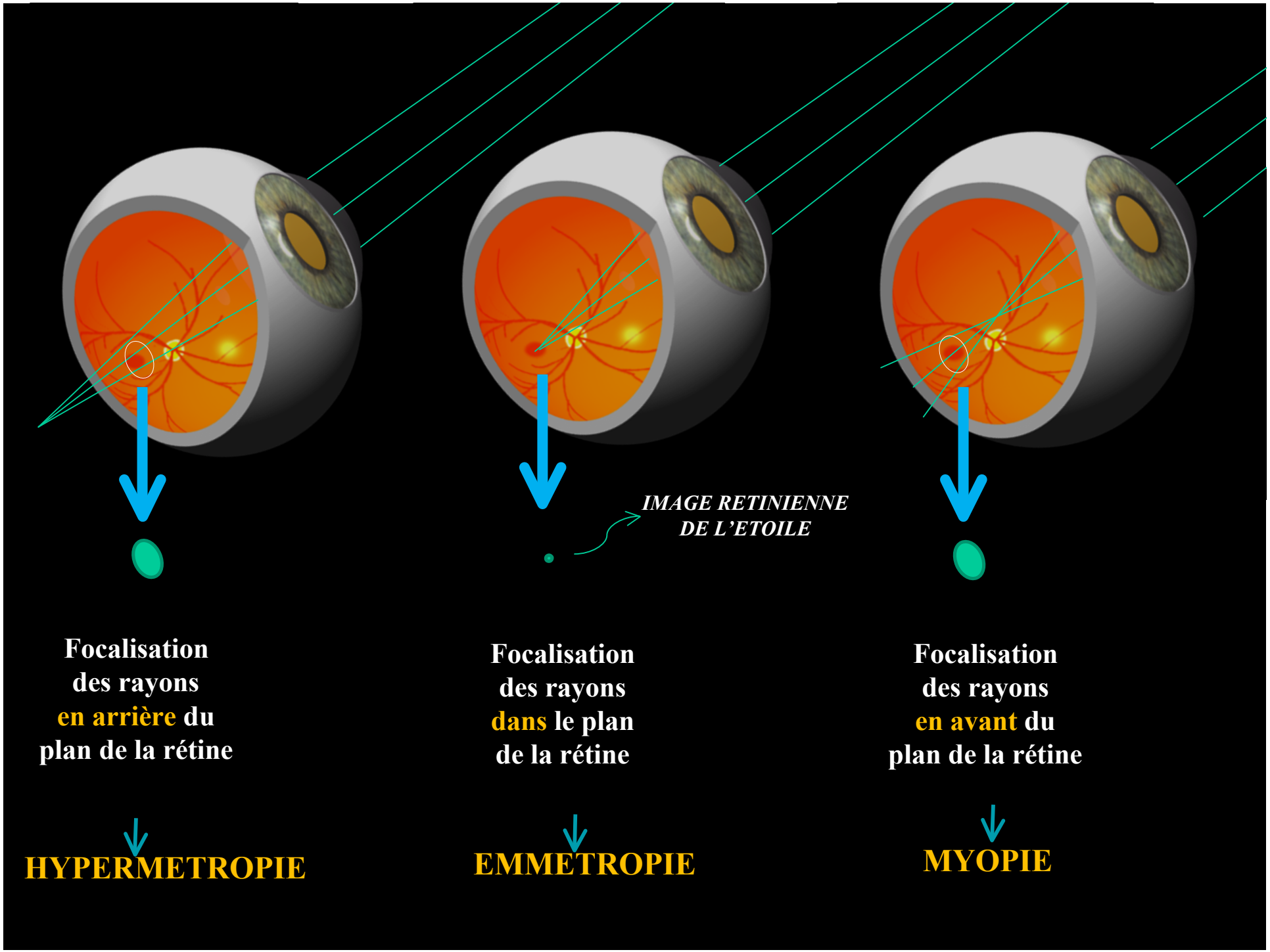
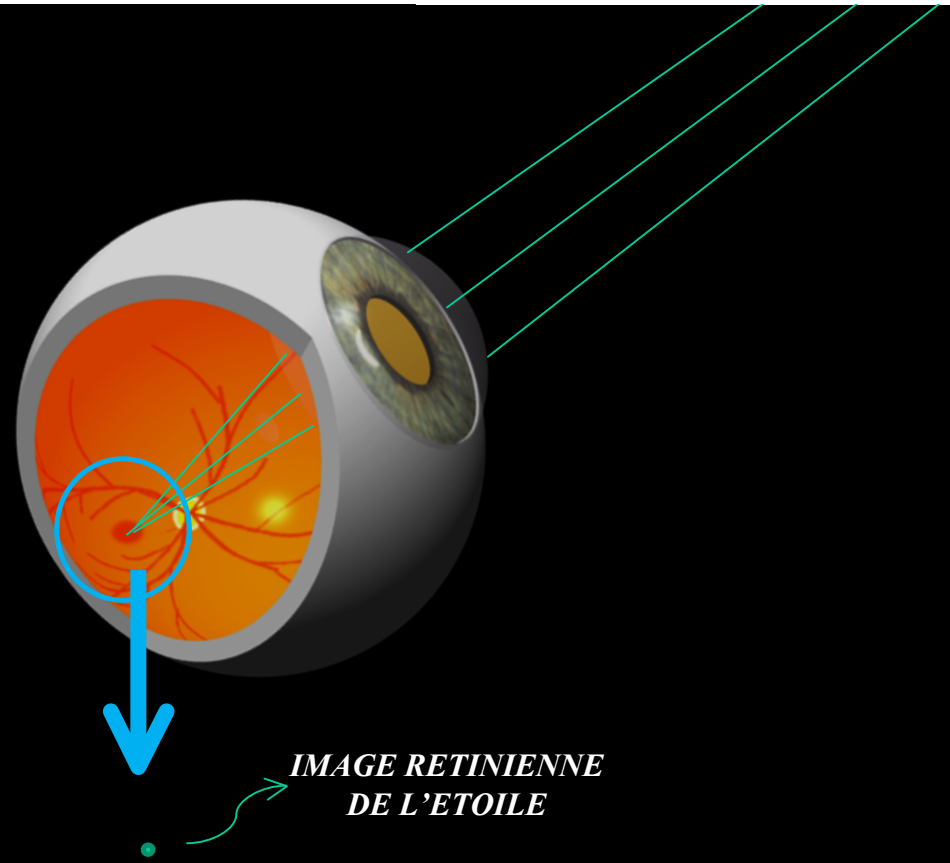


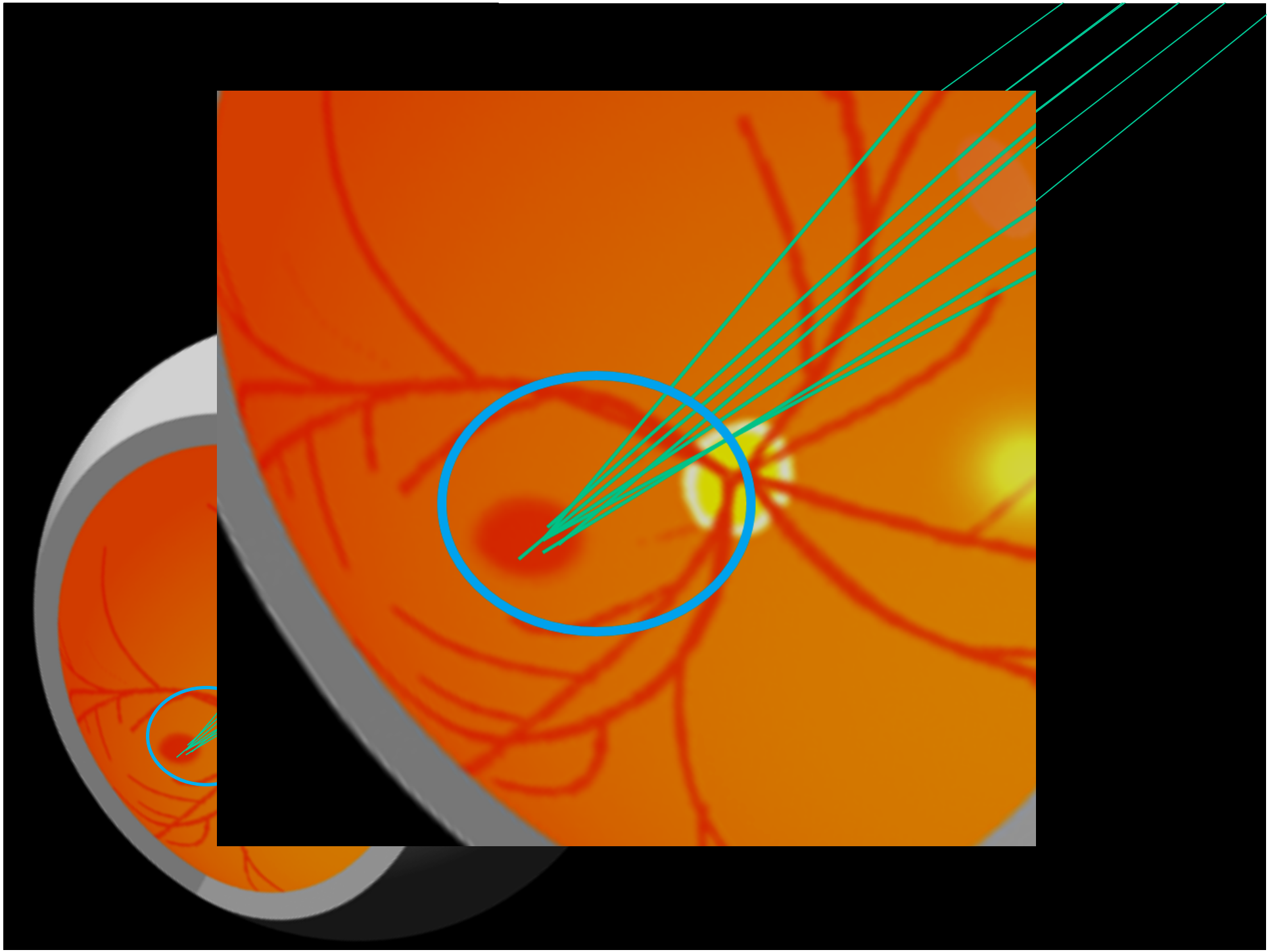
IMAGE RETINIENNE
DE L'ETOILE
Focalisation
des rayons
en arrière du
plan de la rétine
HYPERMETROPIE
Focalisation
des rayons
dans le plan
de la rétine
EMMETROPIE
Focalisation
des rayons
en avant du
plan de la rétine
MYOPIE

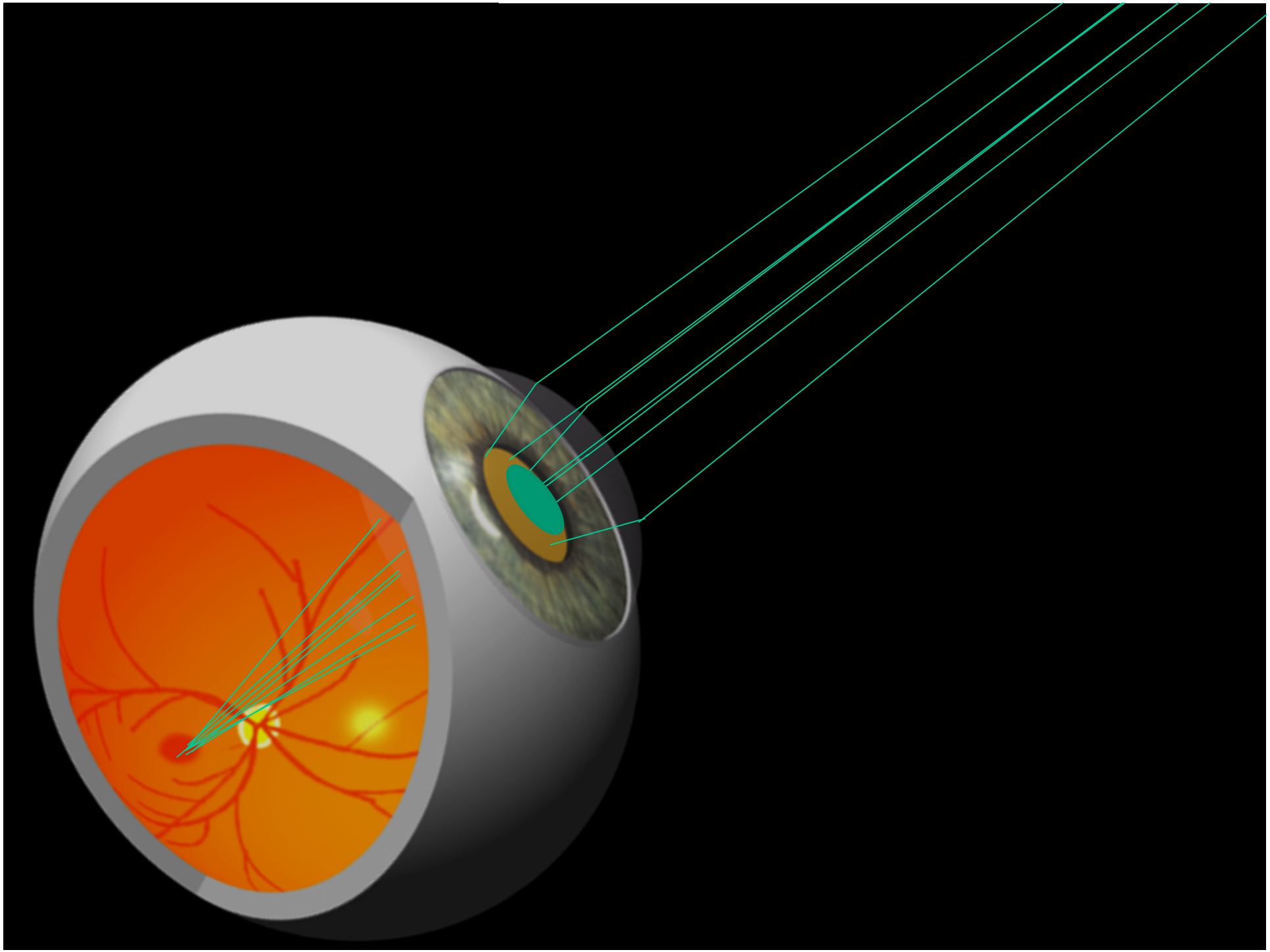


Focalisation
des rayons
dans le plan
de la rétine

↓

EMMETROPIE





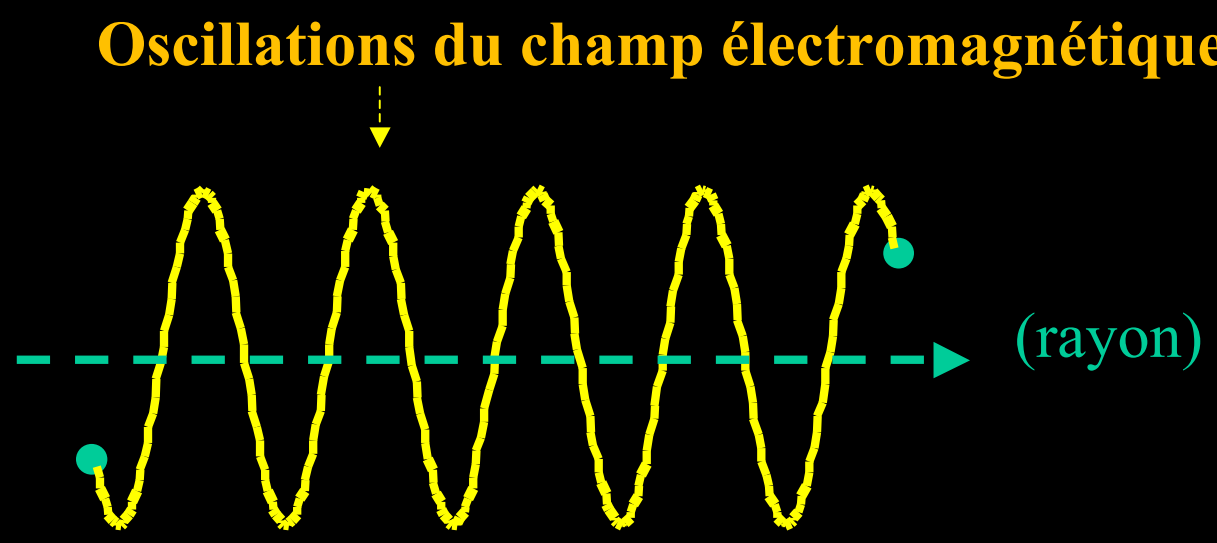
ETUDE DE LA QUALITE OPTIQUE DE L'ŒIL HUMAIN

L'étude de la qualité optique de l'oeil permet de :

- Objectiver et trouver la cause à certains symptomes visuels
- Explorer et réparer la qualité optique de l'oeil humain
- Etablir des normes objectives et quantitatives précises
- Concevoir de nouvelles solutions thérapeutiques

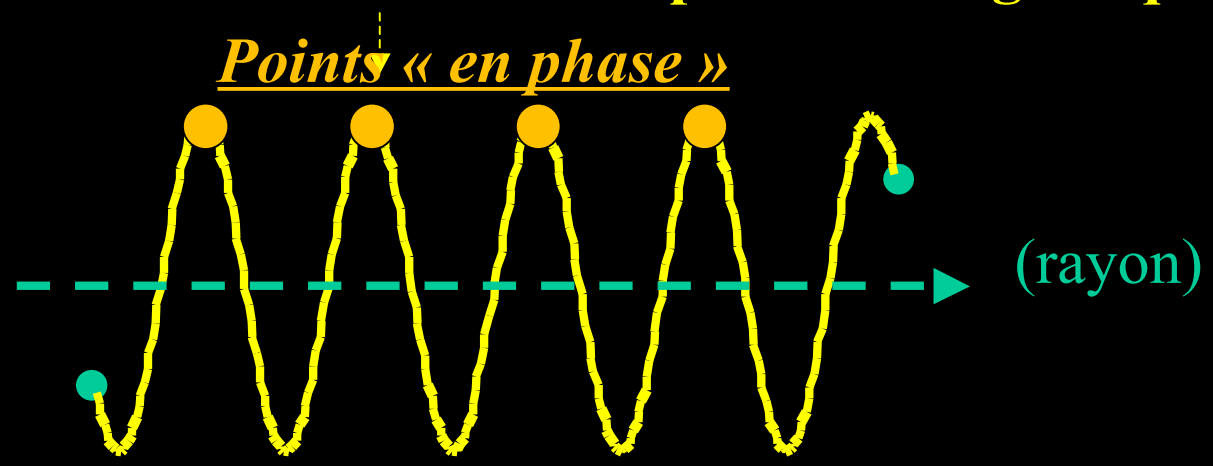
NOTION DE FRONT D’ONDE

Pour comprendre la notion de « front d'onde », il convient d'adopter une conception « ONDULATOIRE » de la lumière.

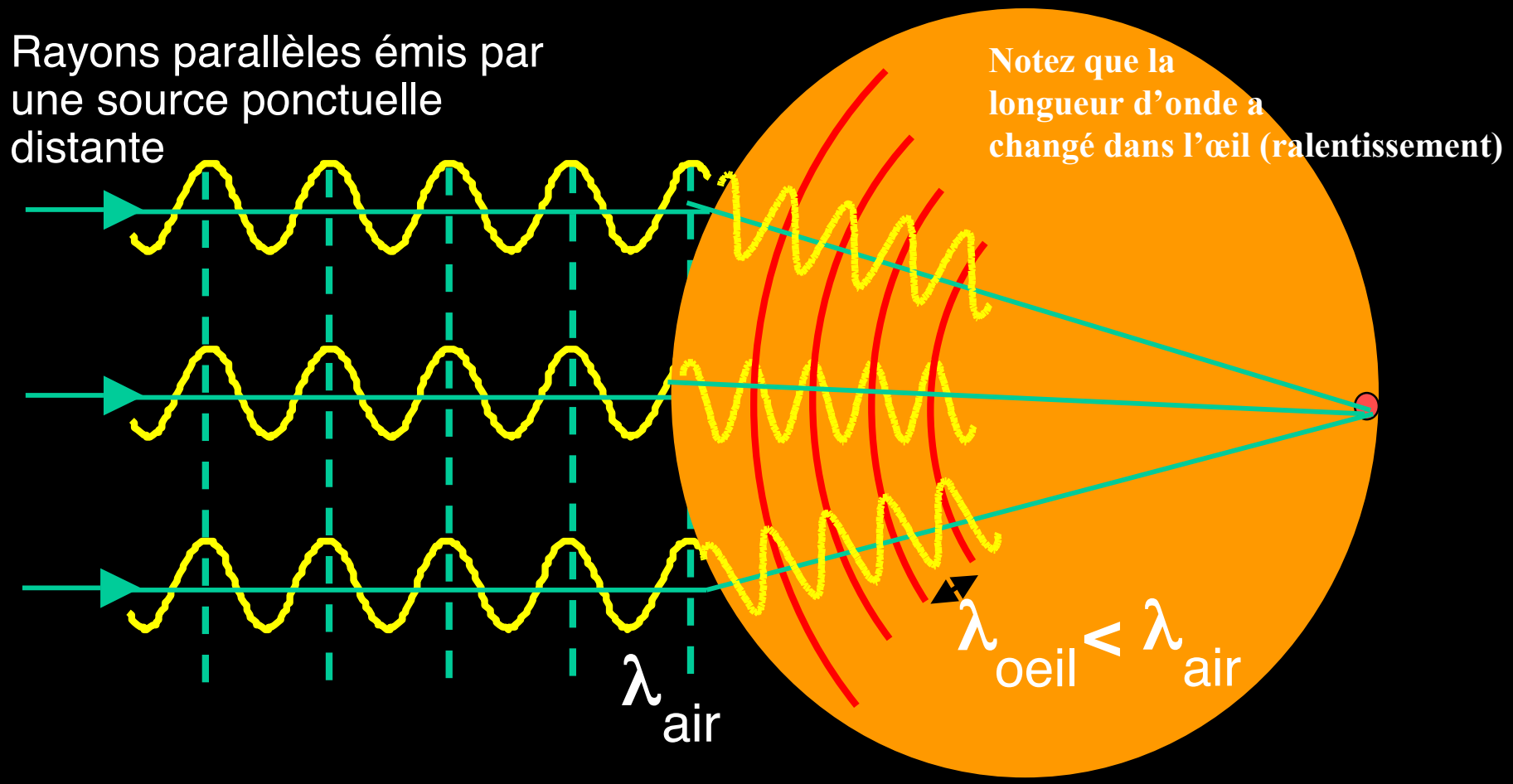


Pour comprendre la notion de « front d'onde », il convient d'adopter une conception « ONDULATOIRE » de la lumière.

Oscillations du champ électromagnétique

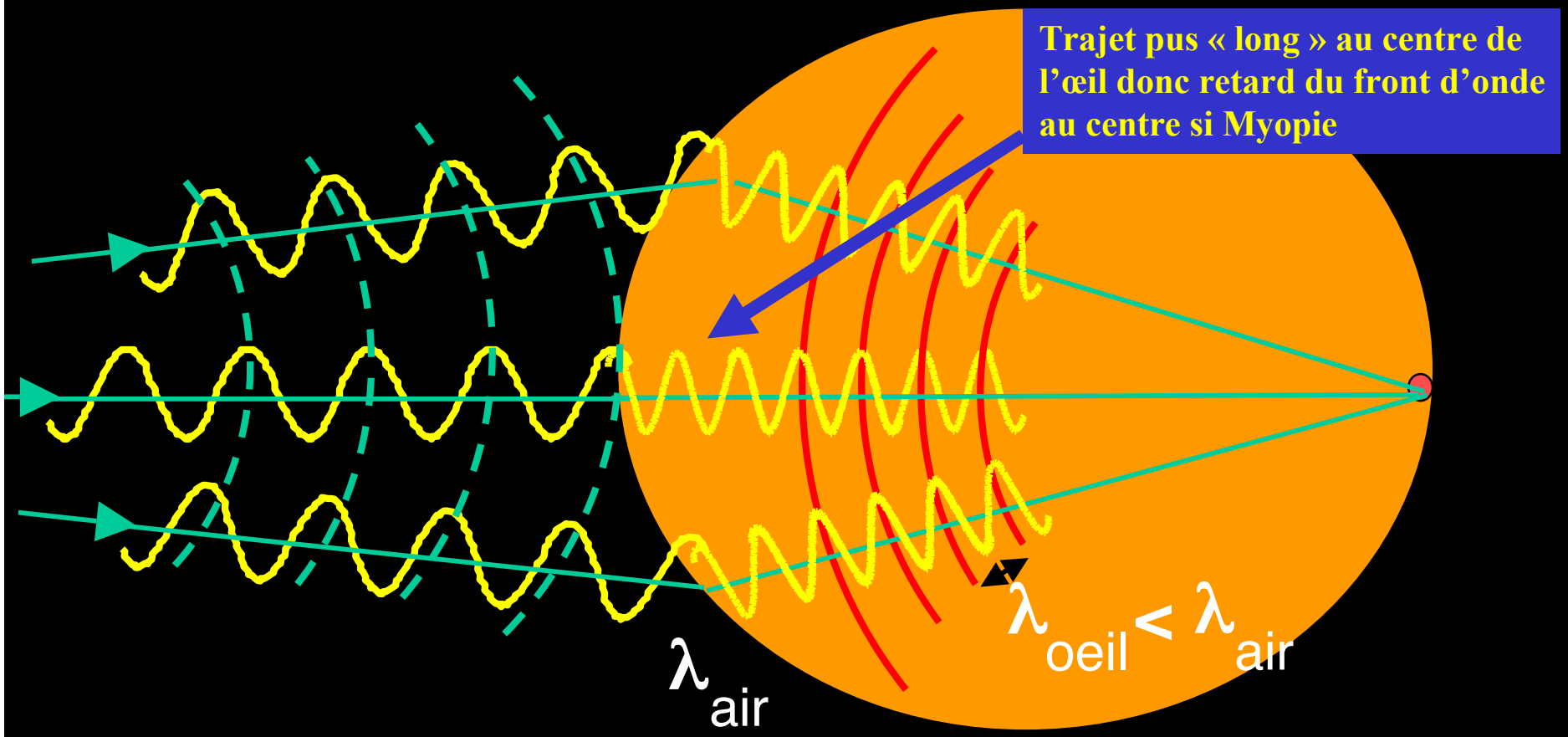


Chemins optiques identiques => images parfaîtes



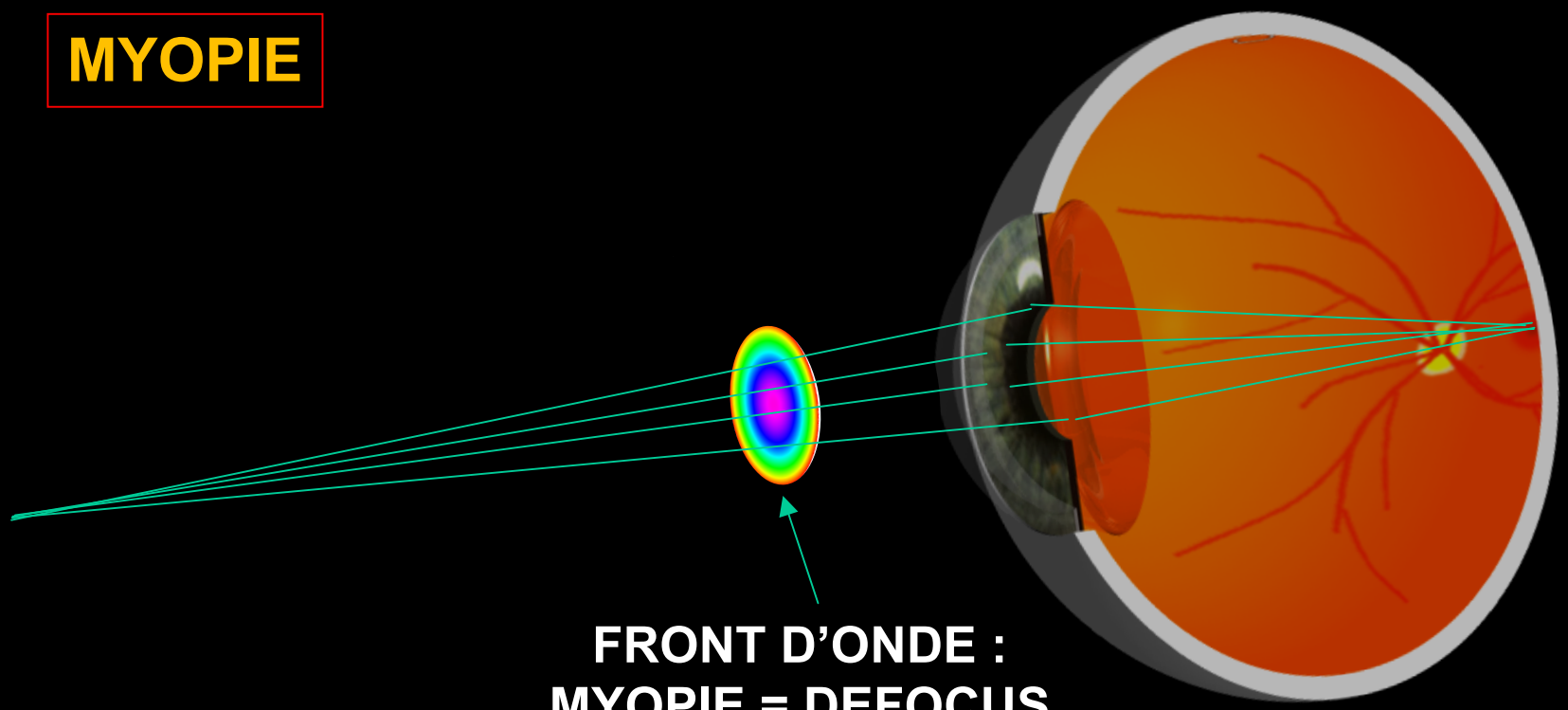
- **Chemin optique égal=> phase égale dans le plan de l'image**
- **Phase égale =>interférence constructive, image parfaîte**

Chemins optiques DIFFERENTS => images distordues

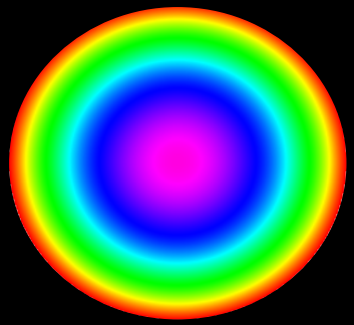


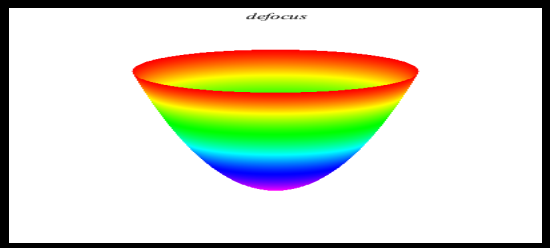
OEIL MYOPE

MYOPIE

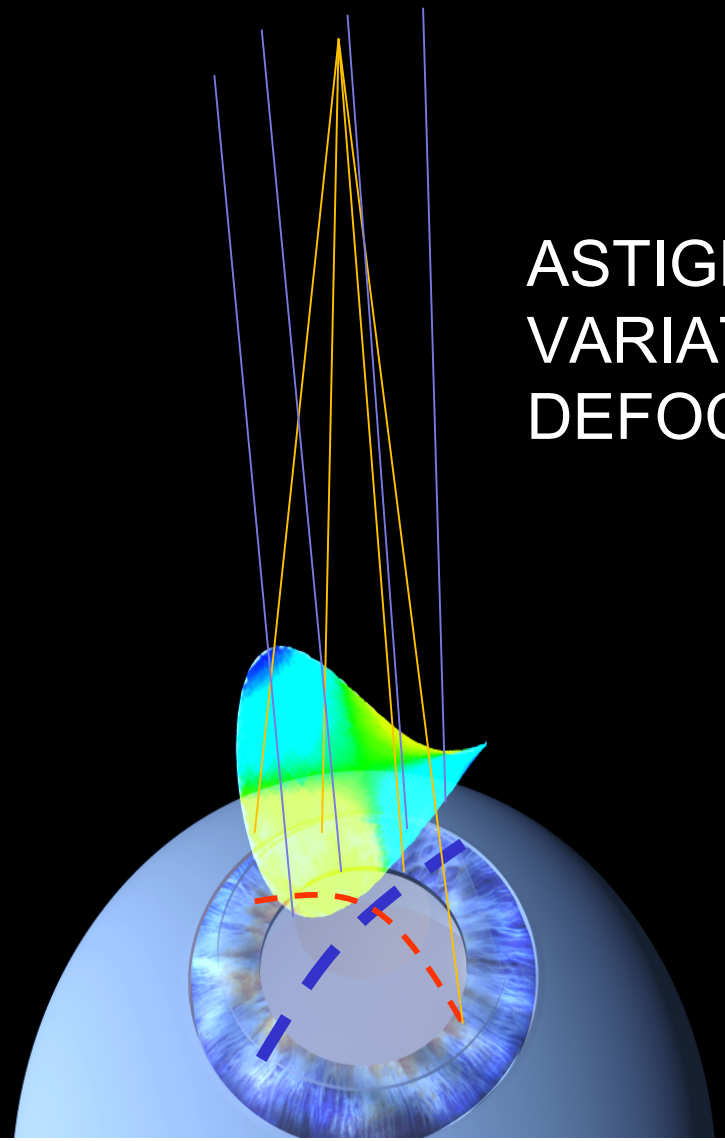


FRONT D'ONDE :
MYOPIE = DEFOCUS
= FONCTION QUADRATIQUE

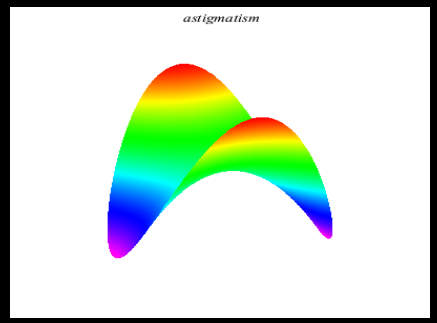




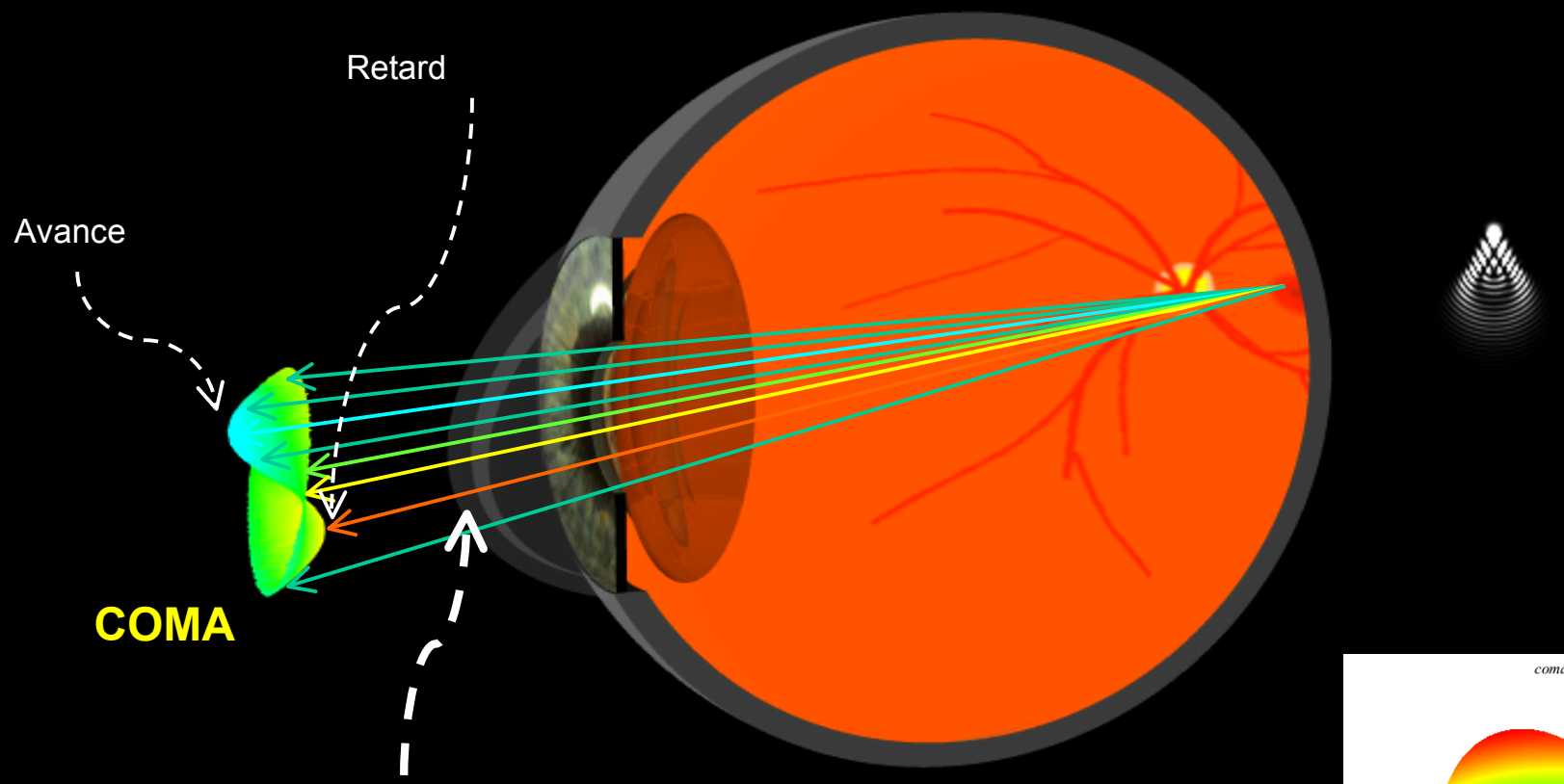
ASTIGMATISME



ASTIGMATISME
VARIATION AZIMUTALE DU DEFOCUS

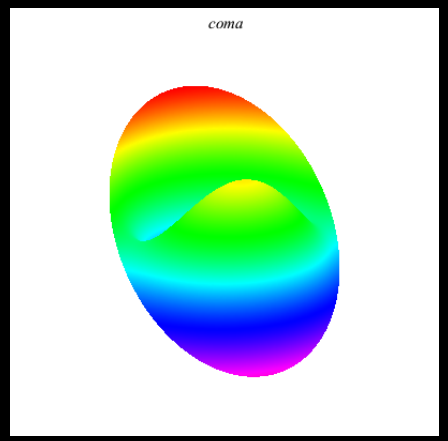


ABERRATION DE COMA

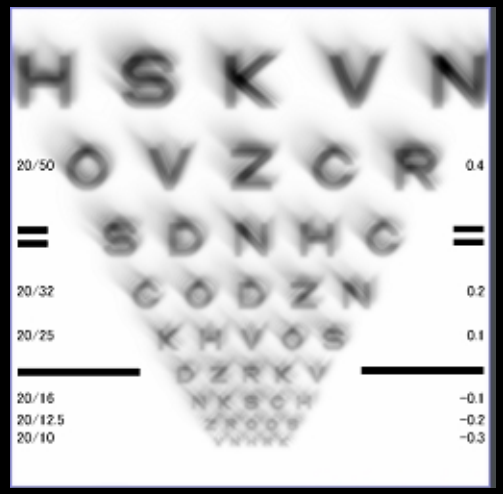


Corneal deformation

COMA = FONCTION CUBIQUE



SYMPTOMES VISUELS LIES A L'ERREUR DU FRONT D'ONDE NON CORRIGIBLE EN LUNETTES



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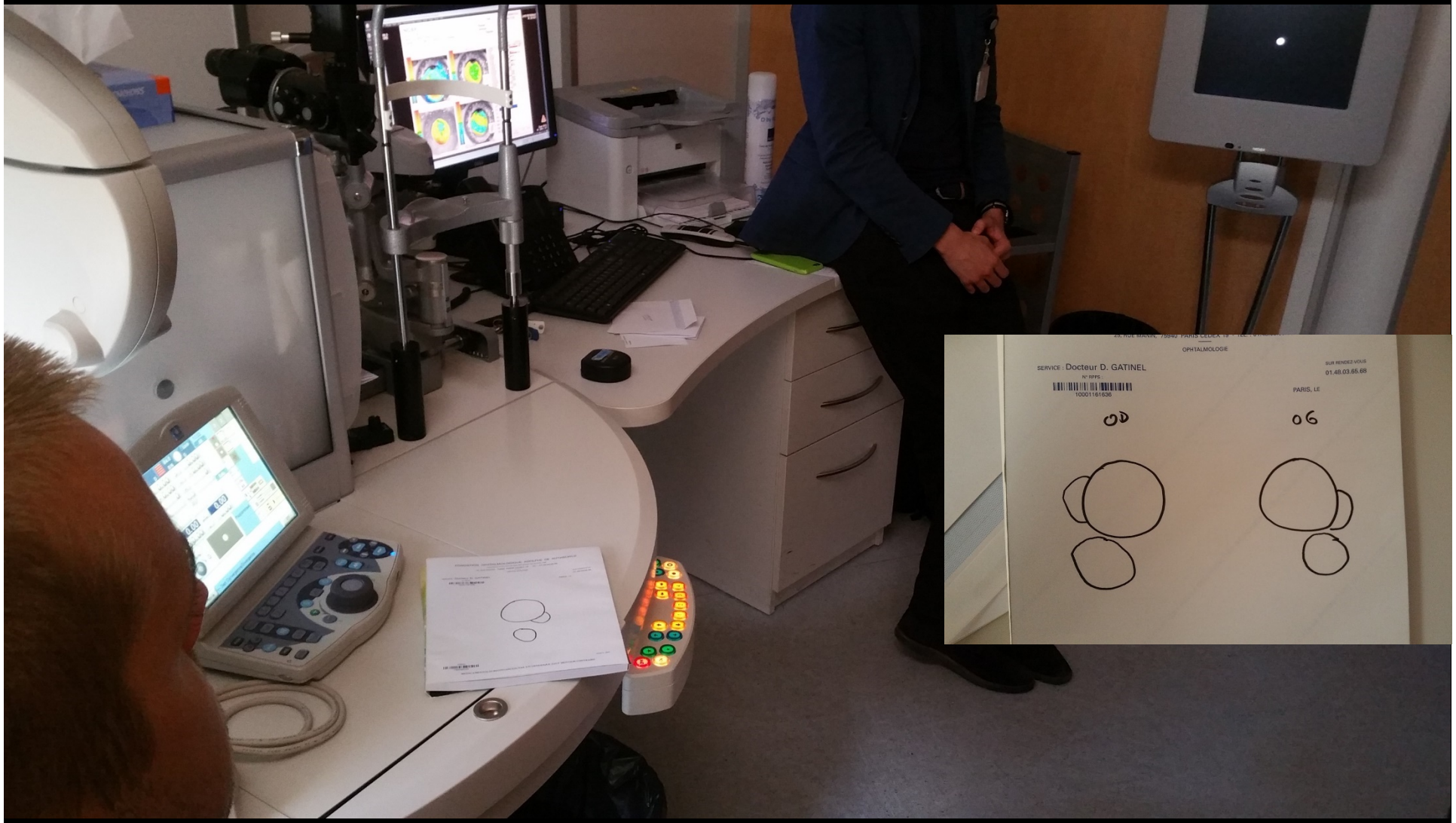
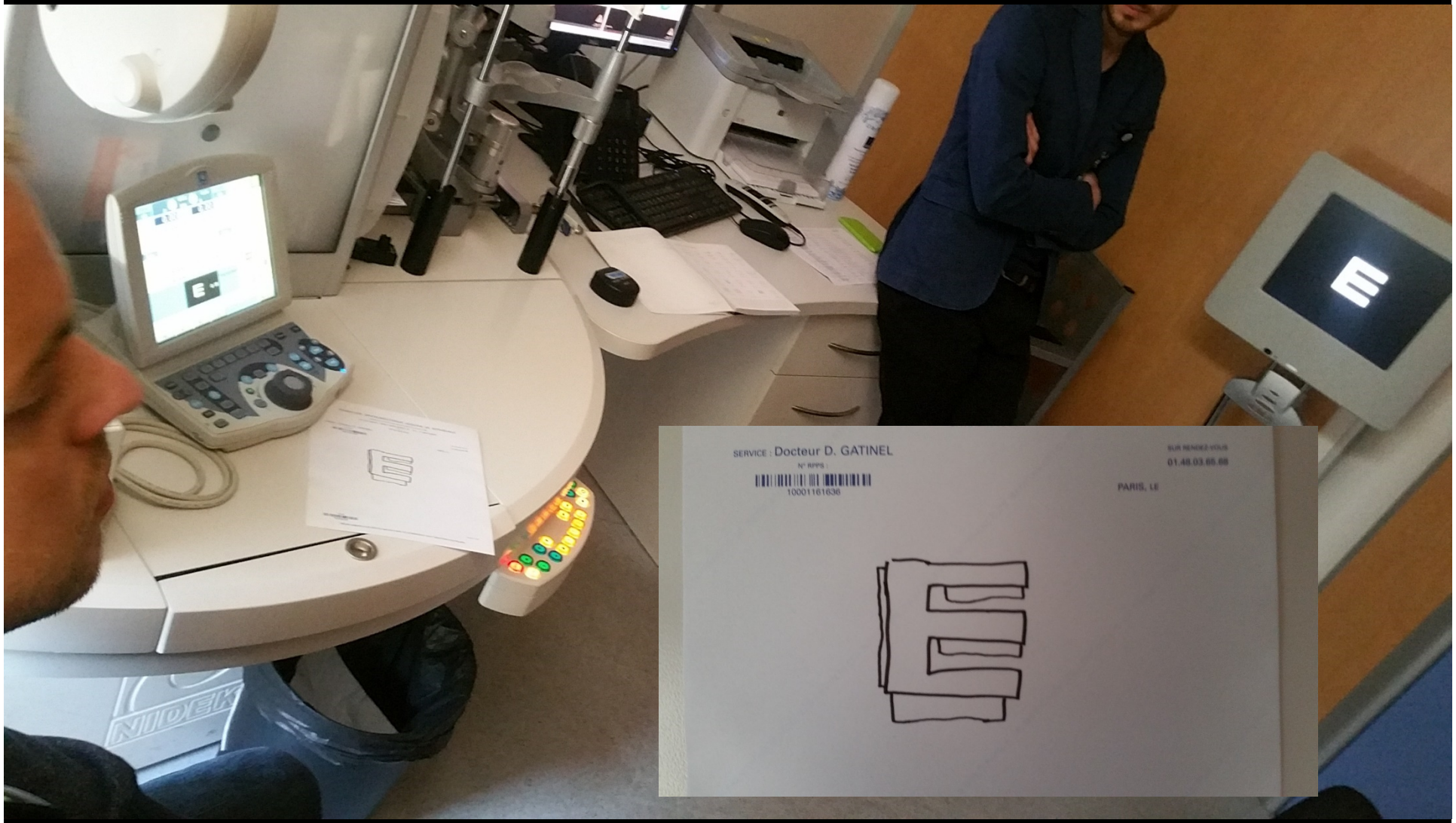
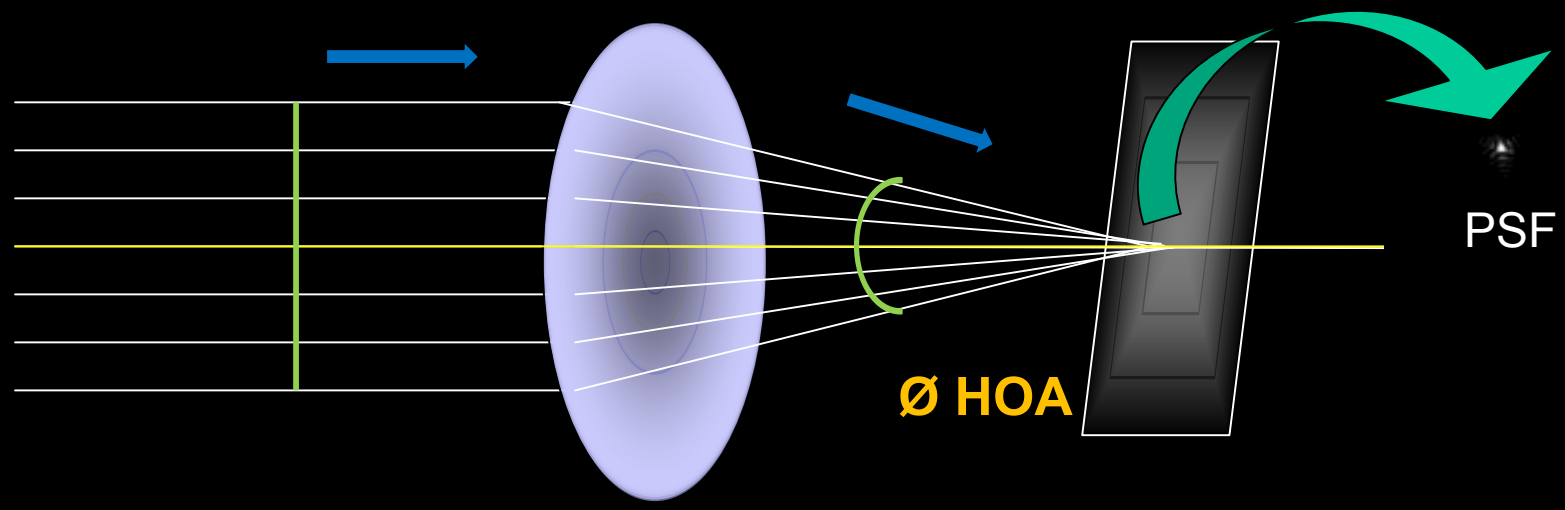


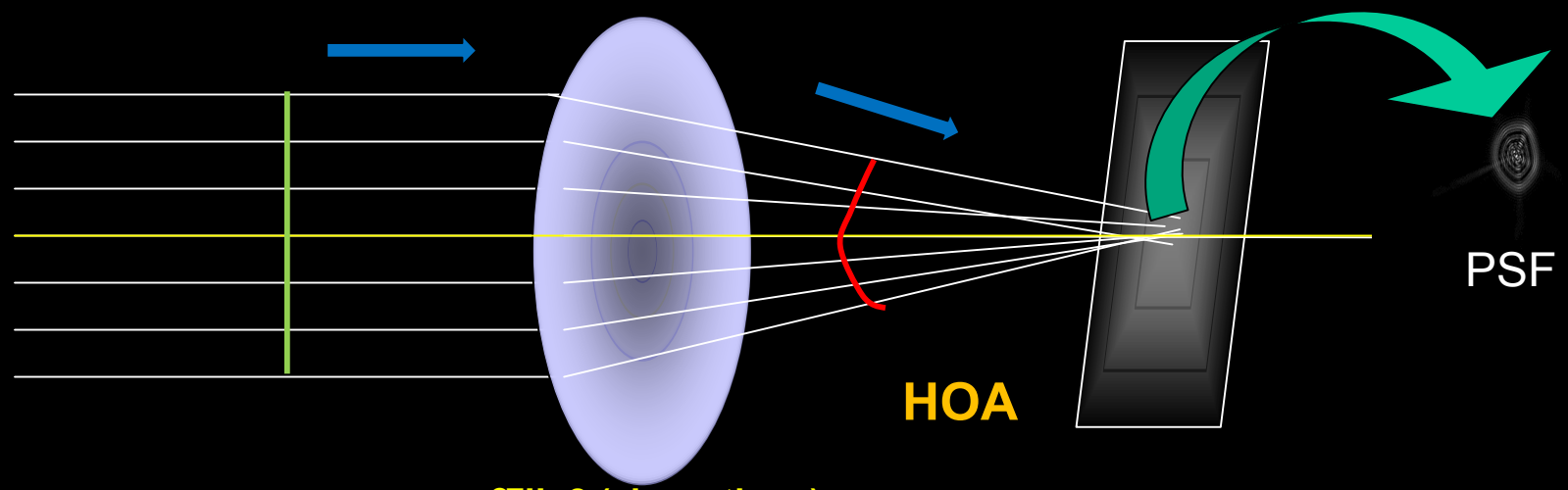
SYMPTOMES VISUELS LIES A L'ERREUR DU FRONT D'ONDE NON CORRIGIBLE EN LUNETTES



EN RESUME :



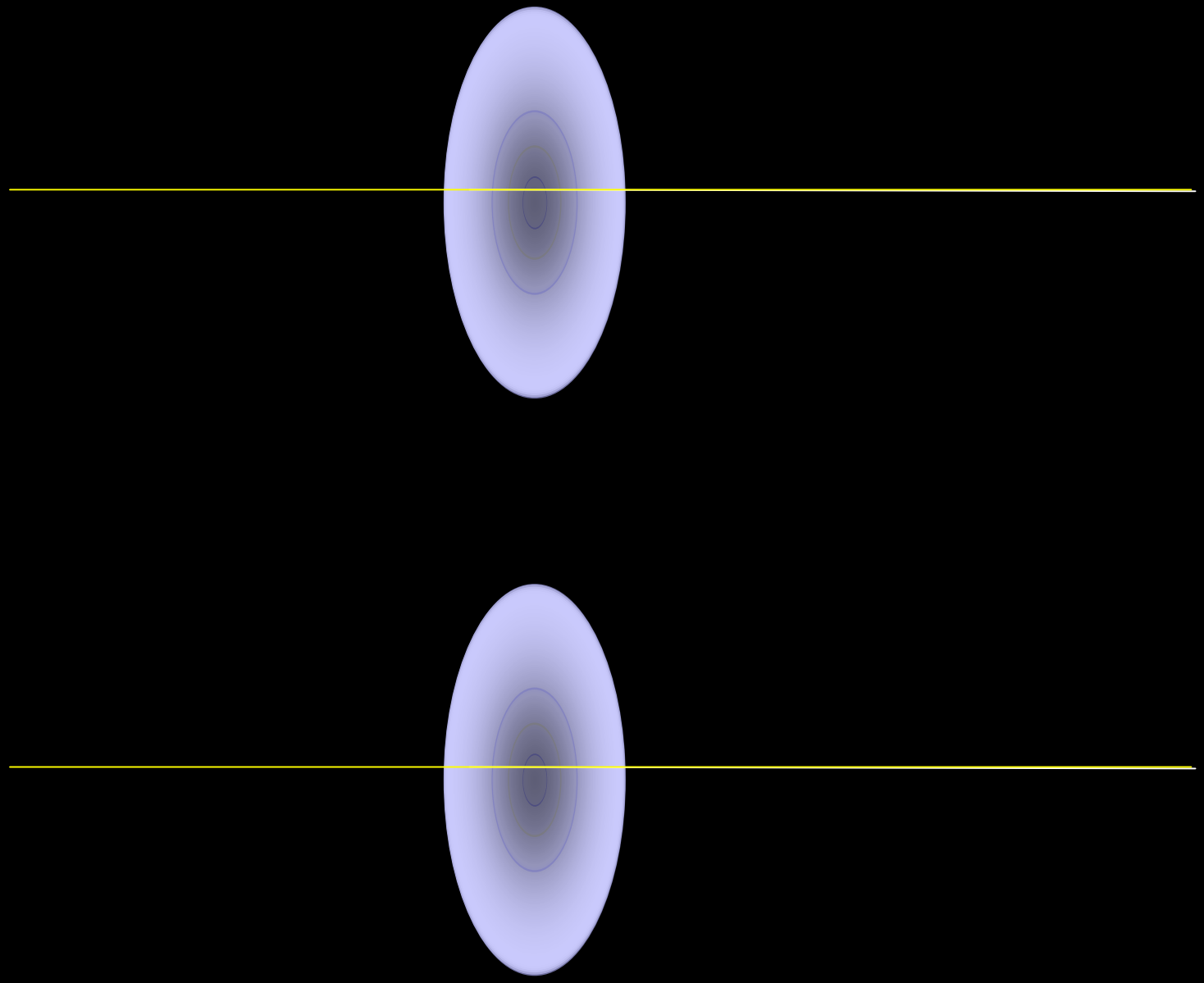
OEIL 1 (pas d'aberrations)



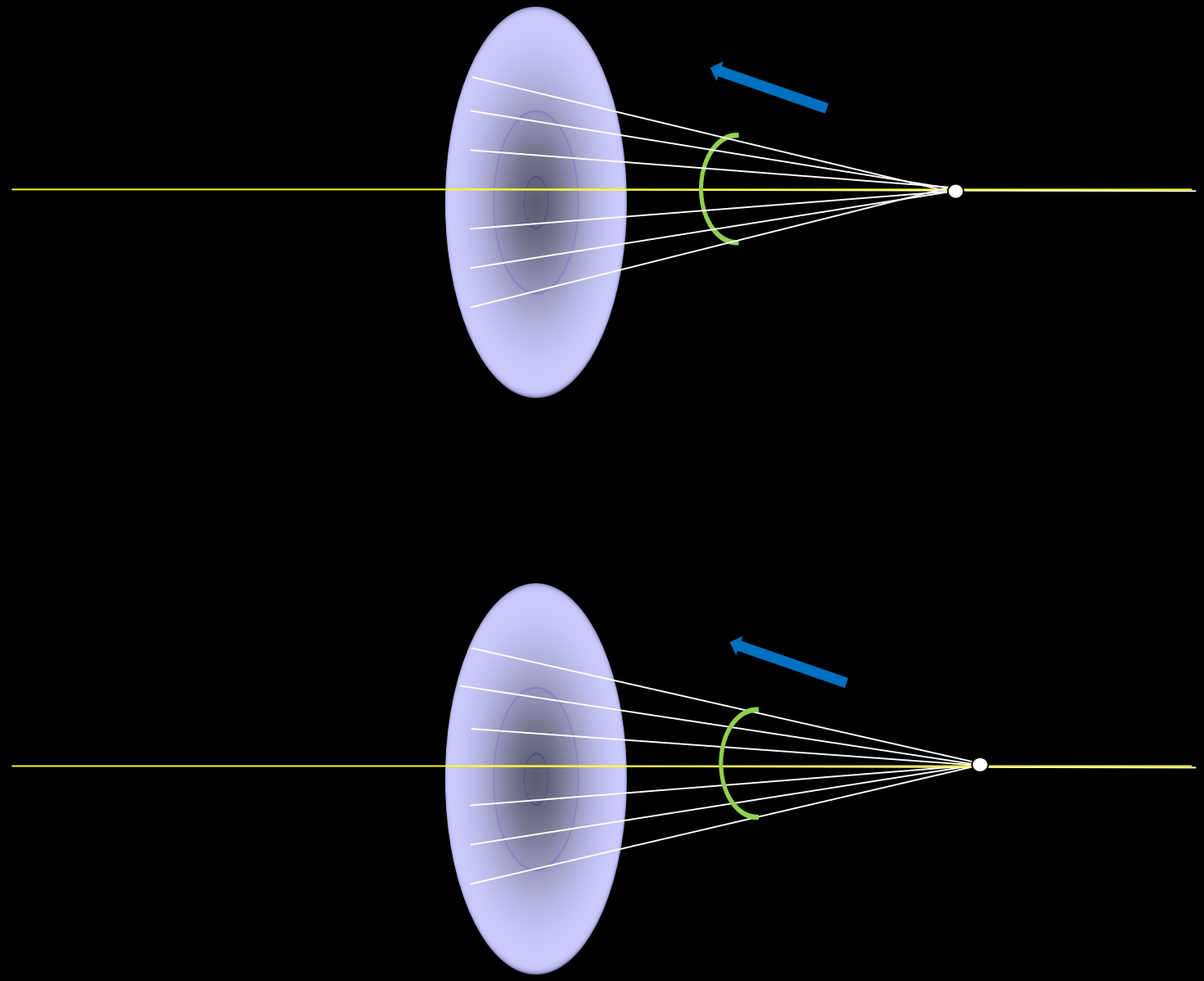
ŒIL 2 (aberrations)

RECUEIL DU FRONT D’ONDE

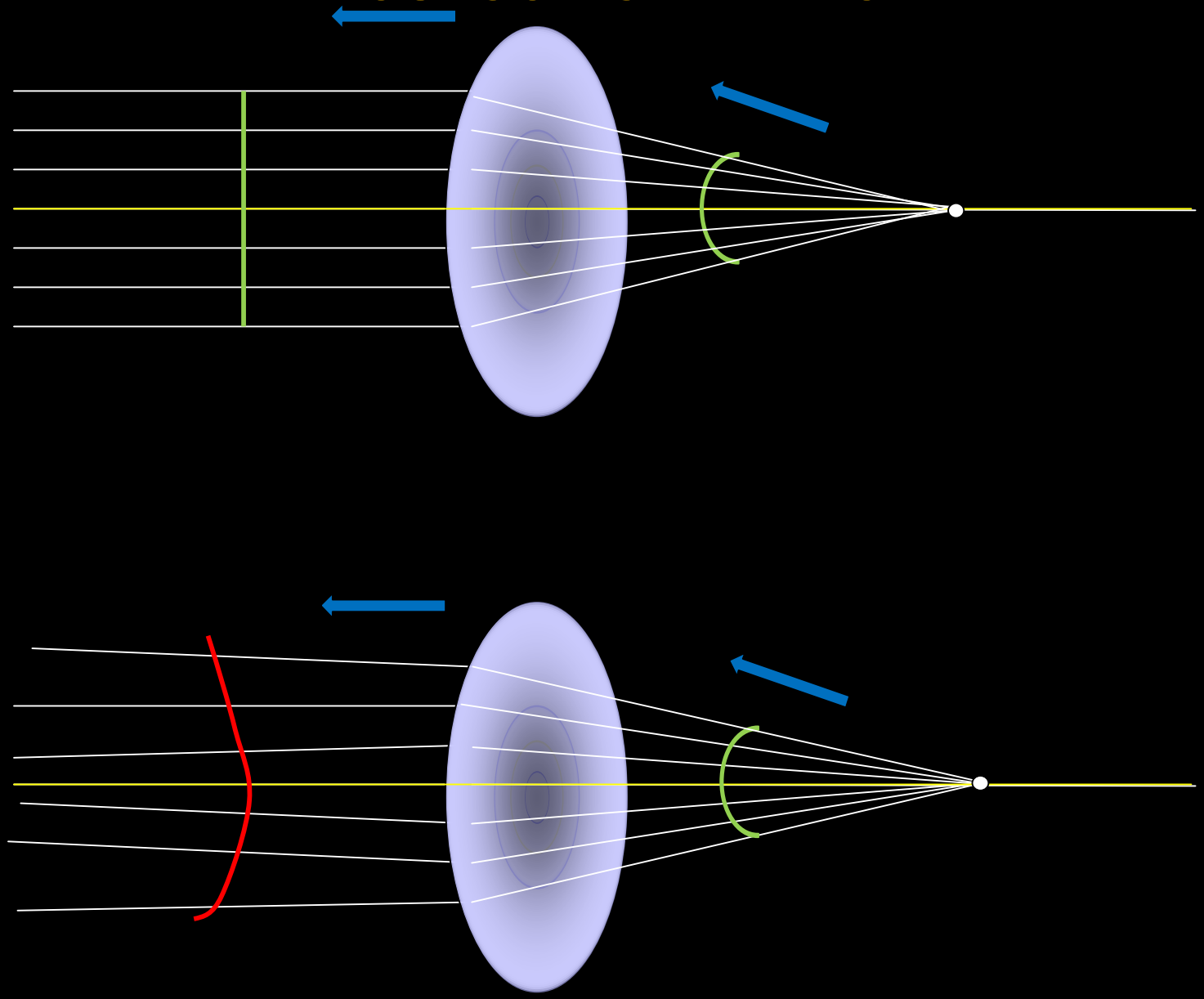
« OUTGOING ABERROMETRY »



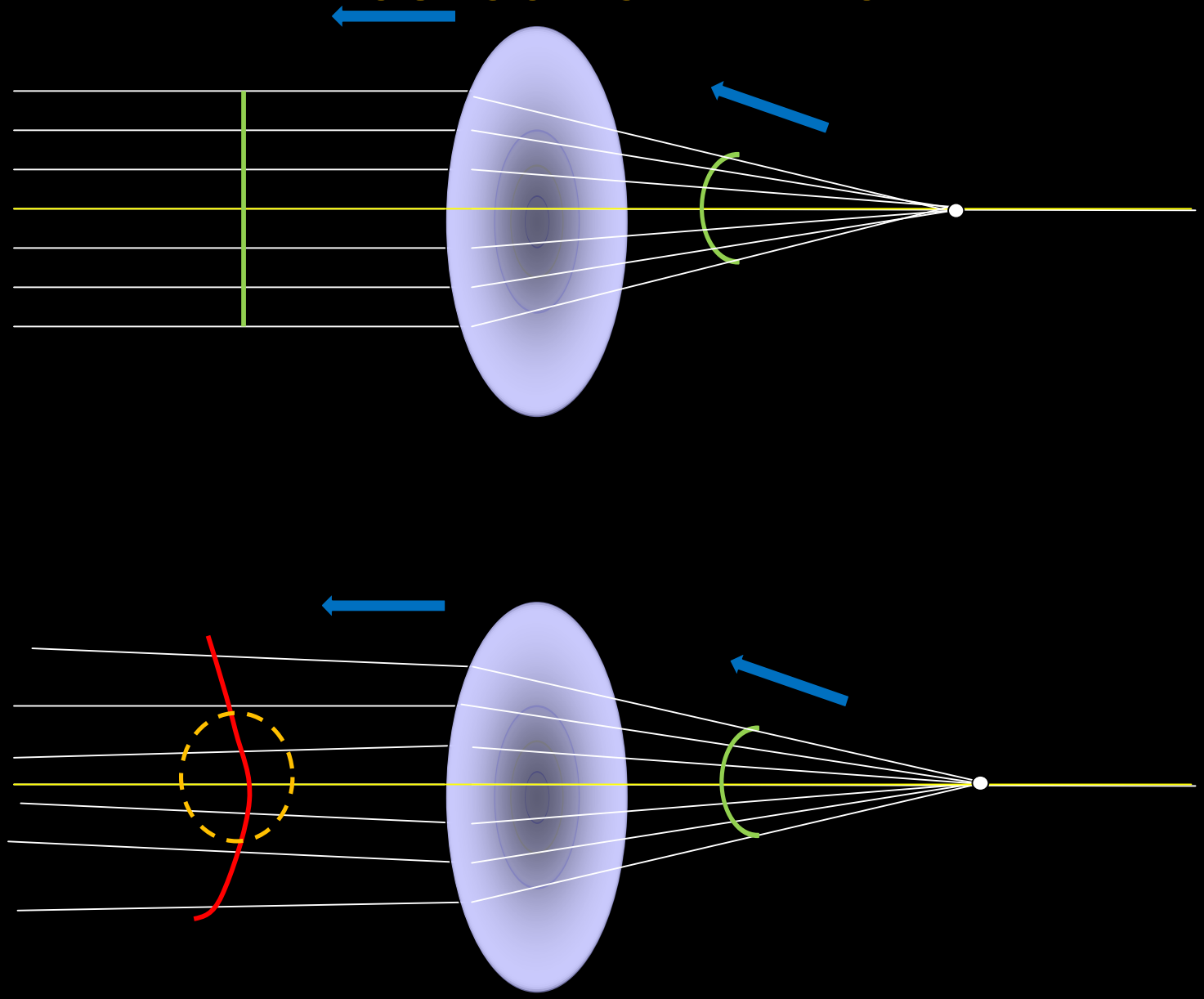
« OUTGOING ABERROMETRY »



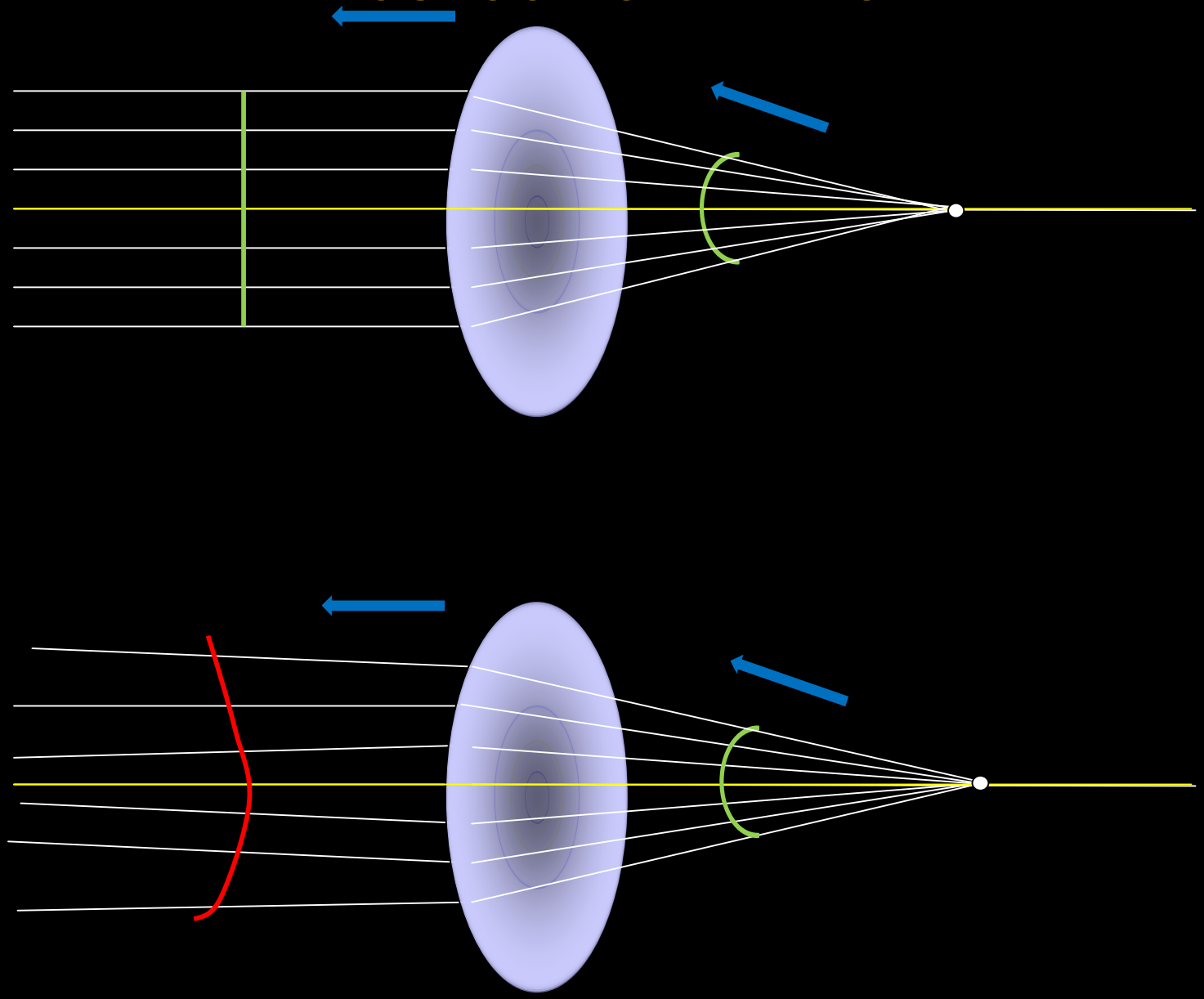
« OUTGOING ABERROMETRY »



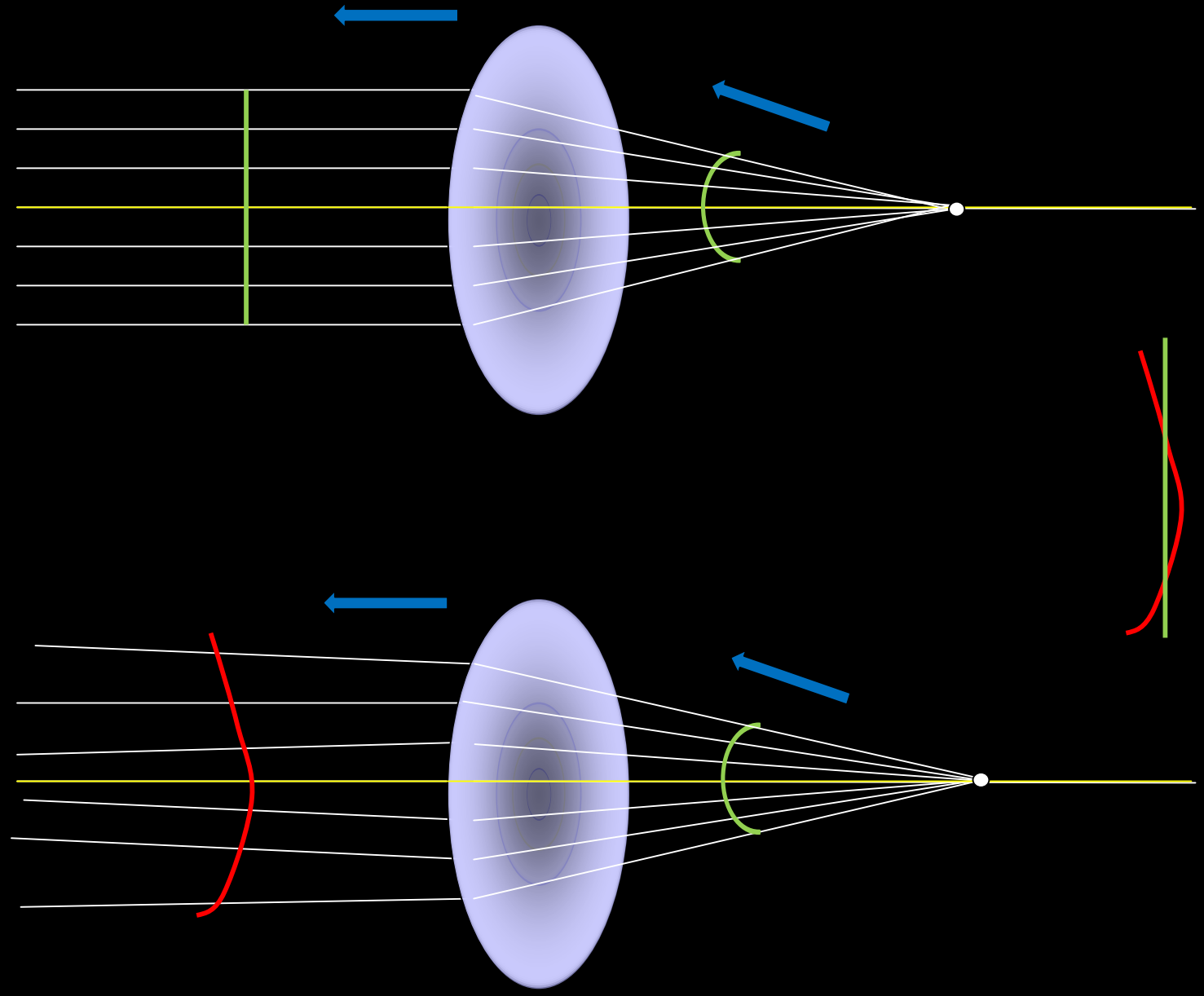
« OUTGOING ABERROMETRY »



« OUTGOING ABERROMETRY »

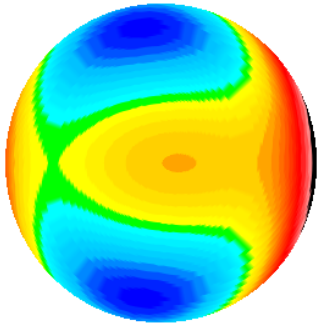


« OUTGOING ABERROMETRY »



Partie 2: traduction mathématique

Analyse mathématique du front d'onde

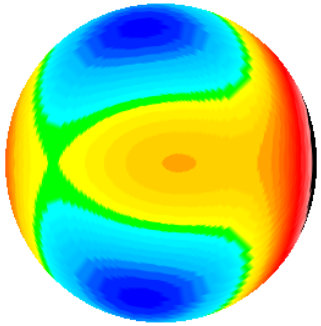


$$f : \begin{cases} \mathbf{D} \longrightarrow \mathbb{R} \\ (x, y) \longmapsto f(x, y) \end{cases}$$

$$\mathbf{D} = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 \leq 1\}$$

$$< f_1, f_2 >_{\mathbf{D}} = \frac{1}{\pi} \iint_{\mathbf{D}} f_1(x, y) f_2(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

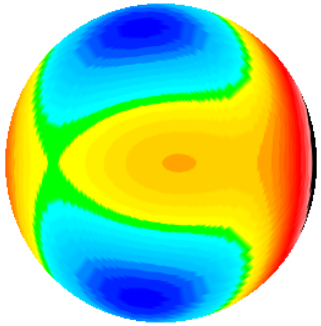
Analyse mathématique du front d'onde



$$p_0(f) = \frac{1}{\pi} \int_{\mathbf{D}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$V(f) = \parallel f - p_0(f) \parallel_{\mathbf{D}}{}^2 = \frac{1}{\pi} \int_{\mathbf{D}} \big(f(x,y) - p_0(f)\big)^2\,\mathrm{d}x\,\mathrm{d}y$$

Approximation polynomiale du front d'onde



$$\forall (x,y) \in \mathbf{D},\ f(x,y) = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j}\, x^i y^j$$

$$\begin{aligned}
(X_0^0)(x,y) &= 1 \\
(X_1^{-1})(x,y) &= y \\
(X_1^1)(x,y) &= x \\
(X_2^{-2})(x,y) &= y^2 \\
(X_2^0)(x,y) &= xy \\
(X_2^2)(x,y) &= x^2\ .
\end{aligned}$$

Approximation polynomiale du front d'onde

$$f = \sum_{n=0}^{\infty} \Big(\sum_{m \in I_n} x_n^m(f)\, X_n^m \Big)$$

	m=-6	m=-5	m=-4	m=-3	m=-2	m=-1	m=0	m=1	m=2	m=3	m=4	m=5	m=6
n=0							X_0^0						
n=1						X_1^{-1}		X_1^{+1}					
n=2					X_2^{-2}		X_2^0		X_2^{+2}				
n=3				X_3^{-3}		X_3^{-1}		X_3^{+1}		X_3^{+3}			
n=4			X_4^{-4}		X_4^{-2}		X_4^0		X_4^{+2}		X_4^{+4}		
n=5		X_5^{-5}		X_5^{-3}		X_5^{-1}		X_5^{+1}		X_5^{+3}		X_5^{+5}	
n=6	X_6^{-6}		X_6^{-4}		X_4^{-2}		X_6^0		X_6^{+2}		X_6^{+4}		X_6^{+6}

Tableau 2 *la pyramide des* X_n^m *jusqu'au degré* 6

Ecriture sous forme trigonométrique

Passage en coordonnées polaires:

$$(x, y) = (r\cos t, r\sin t)$$

Fonction de type radiale:

$$\forall\,(r,t) \in\ [0,1] \times \mathbb{R}, \quad f(r\cos t, r\sin t) = R(r)$$

Fonction de type azimutale:

$$\forall\,(r,t) \in\ [0,1] \times \mathbb{R}, \quad f(r\cos t, r\sin t) = T(t)$$

Fonction tensorisée:

$$\forall\,(r,t) \in\ [0,1] \times [-\pi, +\pi], \quad f(r\cos(t), r\sin(t)) = R(r)T(t)$$

Base orthonormale des polynômes trigonométriques de Zernike

$$f = \sum_{n \in \mathbb{N}} \left(\sum_{m \in I_n} < f, Z_n^m >_{\mathbf{D}} \; Z_n^m \right)$$

Fritz ZERNIKE (1888 – 1966)
Nobel : 1953

$$Z_0^0$$

$$Z_1^{-1} \quad Z_1^{+1}$$

$$Z_2^{-2} \quad Z_2^0 \quad Z_2^{+2}$$

$$Z_3^{-3} \quad Z_3^{-1} \quad Z_3^{+1} \quad Z_3^{+3}$$

$$Z_4^{-4} \quad Z_4^{-2} \quad Z_4^0 \quad Z_4^{+2} \quad Z_4^{+4}$$

$$Z_5^{-5} \quad Z_5^{-3} \quad Z_5^{-1} \quad Z_5^{+1} \quad Z_5^{+3} \quad Z_5^{+5}$$

$$Z_6^{-6} \quad Z_6^{-4} \quad Z_4^{-2} \quad Z_6^0 \quad Z_6^{+2} \quad Z_6^{+4} \quad Z_6^{+6}$$

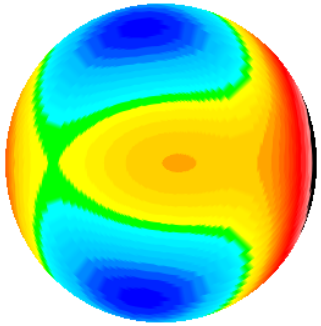
Tableau 4 *la pyramide des polynômes de Zernike* Z_n^m *jusqu'au degré* 6

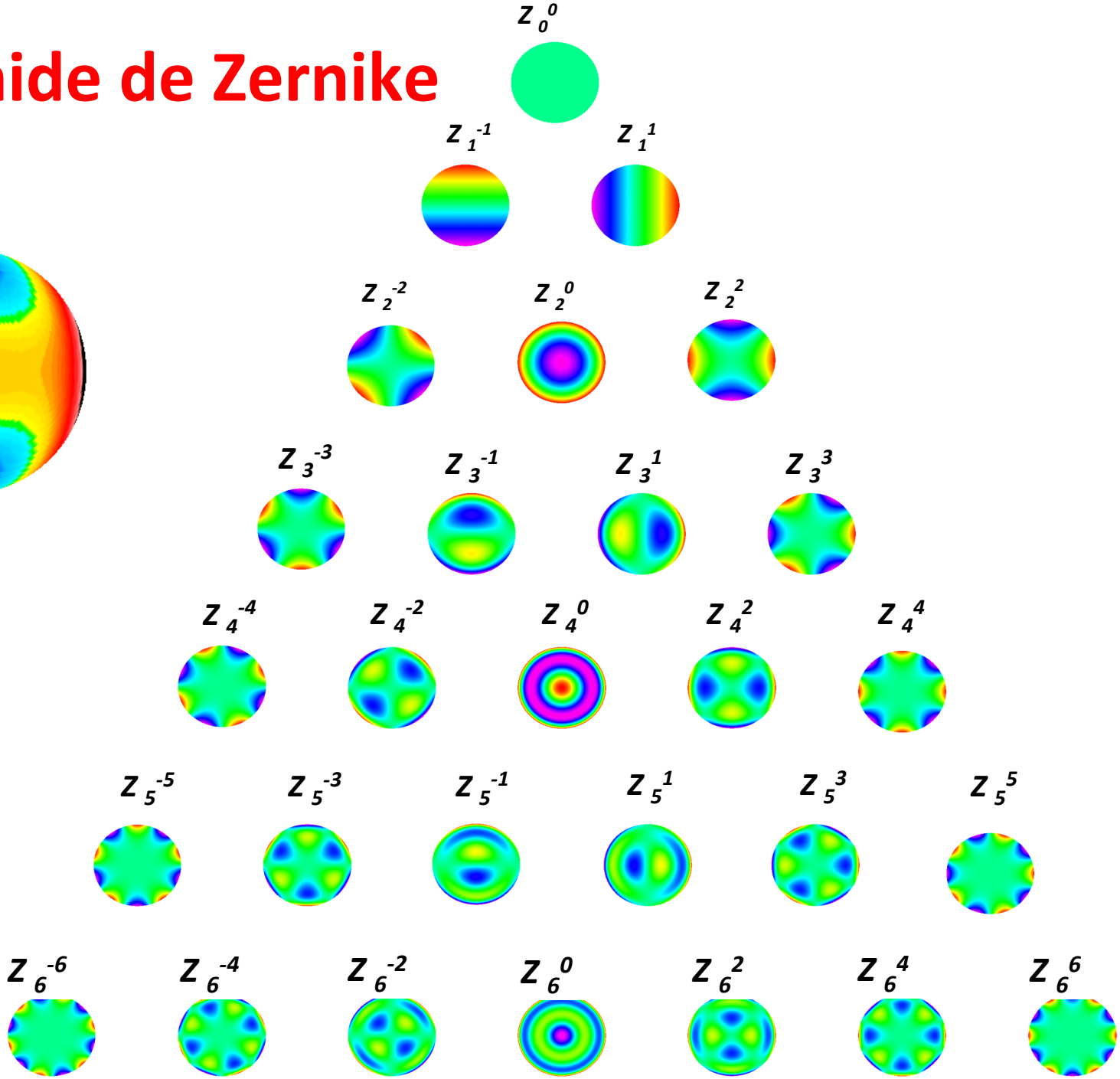
Base orthogonale des polynômes trigonométriques de Zernike

Facteur de normalisation — *Fonctions radiale* — *Fonction azimutale*

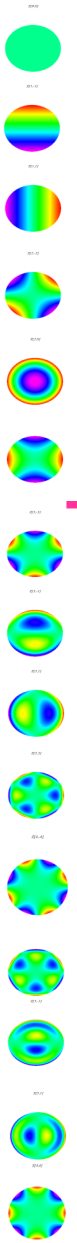
$$Z_n^m = \begin{cases} \sqrt{2(n+1)} \displaystyle\sum_{0 \le k \le \frac{n-m}{2}} (-1)^k \frac{(n-k)!}{k!\left(\frac{n-m}{2}-k\right)!\left(\frac{n+m}{2}-k\right)!} r^{n-2k} \cos(mt) & \text{if } m > 0 \\ \sqrt{n+1} \displaystyle\sum_{0 \le k \le \frac{n}{2}} (-1)^k \frac{(n-k)!}{k!\left(\frac{n}{2}-k\right)!\left(\frac{n}{2}-k\right)!} r^{n-2k} & \text{if } m = 0 \\ \sqrt{2(n+1)} \displaystyle\sum_{0 \le k \le \frac{n+m}{2}} (-1)^k \frac{(n-k)!}{k!\left(\frac{n-m}{2}-k\right)!\left(\frac{n+m}{2}-k\right)!} r^{n-2k} \sin(|m|t) & \text{if } m < 0 \end{cases}$$

Pyramide de Zernike





Troubles de la vision associés



$Z_0^0(r) = 1$ — Piston

$Z_1^{-1}(r\cos(t), r\sin(t)) = 2r\sin(t)$ — Tilt vertical (prisme)

$Z_1^{+1}(r\cos(t), r\sin(t)) = 2r\cos(t)$ — Tilt horizontal (prisme)

$Z_2^{-2}(r\cos(t), r\sin(t)) = \sqrt{6}\, r^2 \sin(2t)$ — Astigmatisme oblique primaire

$Z_2^{0}(r\cos(t), r\sin(t)) = \sqrt{3}\,(2r^2 - 1)$ — Myopie/hypermétropie

$Z_2^{+2}(r\cos(t), r\sin(t)) = \sqrt{6}\, r^2 \cos(2t)$ — Astigmatisme vertical/horizontal

$Z_3^{-3}(r\cos(t), r\sin(t)) = 2\sqrt{2}\, r^3 \sin(3t)$ — Vertical Trefoil

$Z_3^{-1}(r\cos(t), r\sin(t)) = 2\sqrt{2}\,(3r^3 - 2r)\sin(t)$ — Vertical Coma

$Z_3^{+1}(r\cos(t), r\sin(t)) = 2\sqrt{2}\,(3r^3 - 2r)\cos(t)$ — Horizontal Coma

$Z_3^{+3}(r\cos(t), r\sin(t)) = 2\sqrt{2}\, r^3 \cos(3t)$ — Horizontal Trefoil

$Z_4^{-4}(r\cos(t), r\sin(t)) = \sqrt{10}\, r^4 \sin(4t)$ — Oblique Quadrafoil

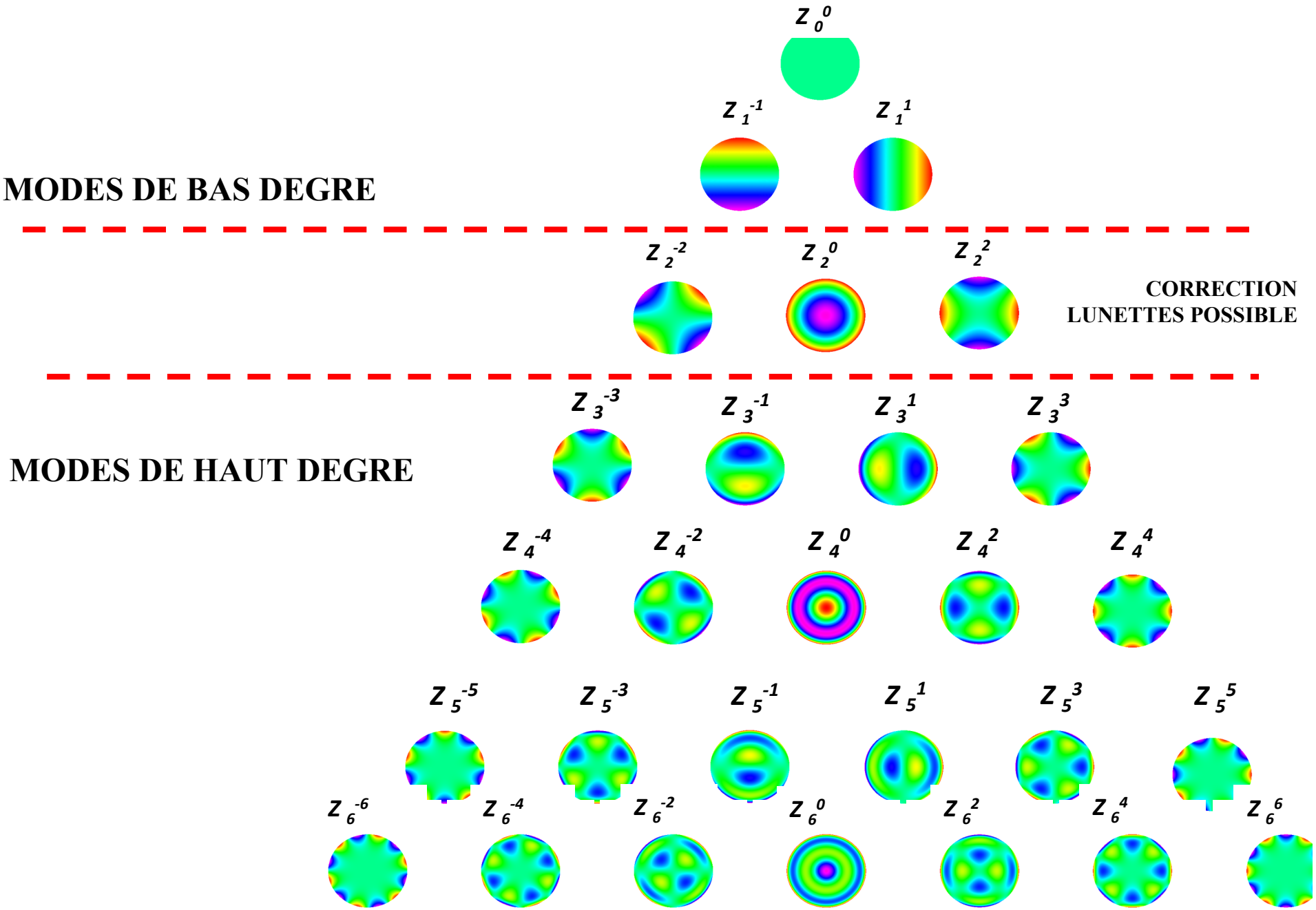
$Z_4^{-2}(r\cos(t), r\sin(t)) = \sqrt{10}\,(4r^4 - 3r^2)\sin(2t)$ — Oblique Secondary Astigmatism

$Z_4^{0}(r\cos(t), r\sin(t)) = \sqrt{5}\,(6r^4 - 6r^2 + 1).$ — Primary Spherical Aberration

$Z_4^{+2}(r\cos(t), r\sin(t)) = \sqrt{10}\,(4r^4 - 3r^2)\cos(2t)$ — Vert/Hor. Secondary Astigmatism

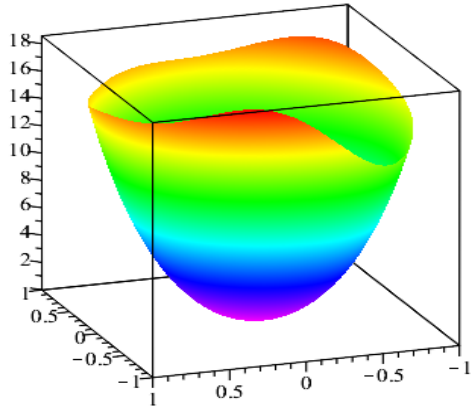
$Z_4^{+4}(r\cos(t), r\sin(t)) = \sqrt{10}\, r^4 \cos(4t)$ — Vertical Quadrafoil

Troubles de la vision associés



Exemple de difficulté de diagnostic

Front d'onde quasi parabolique



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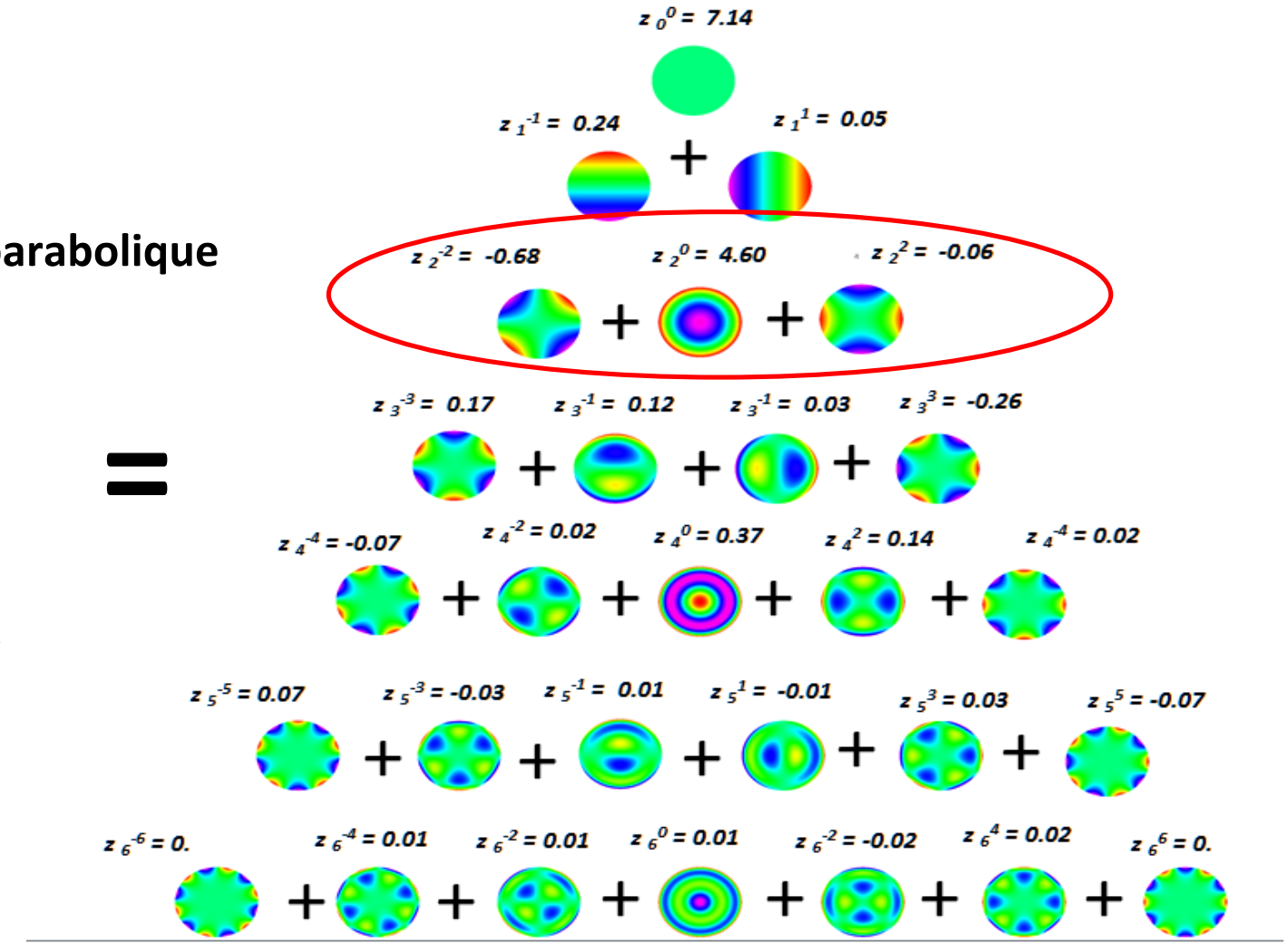


Contamination des termes de bas degré

$$R_0^0(r) = 1$$
$$R_1^1(r) = \boxed{r}$$
$$R_2^0(r) = \boxed{2r^2} - 1$$
$$R_2^2(r) = \boxed{r^2}$$

MODES DE « BAS DEGRE » (DEGRE INF. OU EGAL A 2)

$$R_3^1(r) = 3r^3 - \boxed{2r}$$
$$R_3^3(r) = r^3$$
$$R_4^0(r) = 6r^4 - \boxed{6r^2} + 1$$
$$R_4^2(r) = 4r^4 - \boxed{3r^2}$$
$$R_4^4(r) = r^4$$

MODES DE « HAUT DEGRE »

$$R_5^1(r) = 10r^5 - 12r^3 + \boxed{3r}$$

Termes de degré 1

$$R_5^3(r) = 5r^5 - 4r^3$$
$$R_5^5(r) = r^5$$
$$R_6^0(r) = 20r^6 - 30r^4 + \boxed{12r^2} - 1$$
$$R_6^2(r) = 15r^6 - 20r^4 + \boxed{6r^2}$$
$$R_6^4(r) = 6r^6 - 5r^4$$
$$R_6^6(r) = r^6.$$

Termes de degré 2

Une nouvelle base de décomposition

$$^{(L)}\mathcal{P}_{\mathbf{D}} = \mathrm{Vect}\{\, X_n^m \,;\, n \geq L+1 \,,\, m \in I_n \}$$

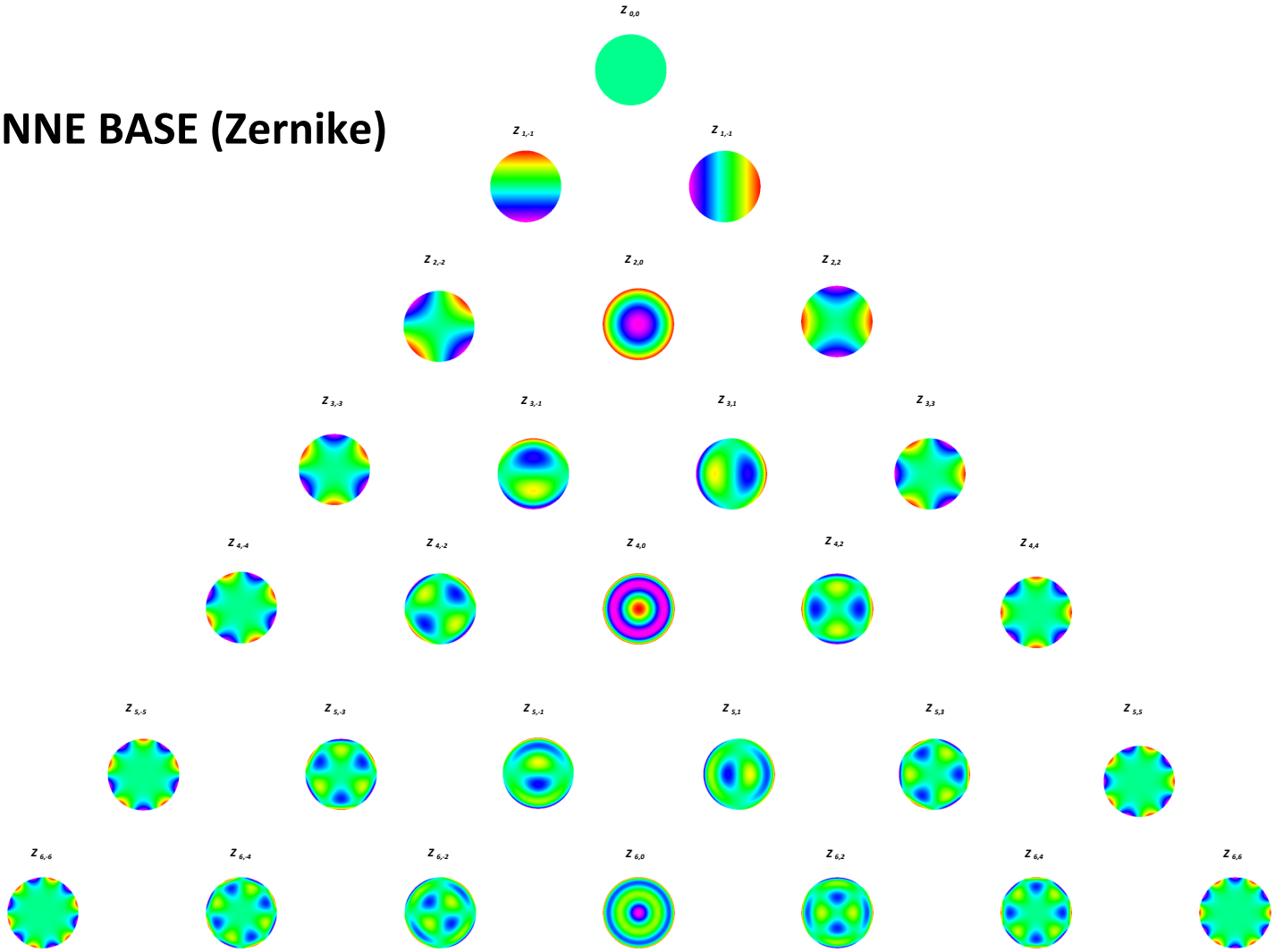
$$\mathcal{P}_{\mathbf{D}} = \mathcal{P}_{\mathbf{D}}^{(L)} \oplus {}^{(L)}\mathcal{P}_{\mathbf{D}}$$

$$f = f_b + f_h$$

$$f_b = \sum_{n=0}^{L} \Big(\sum_{m \in I_n} x_n^m(f)\, X_n^m \Big)$$

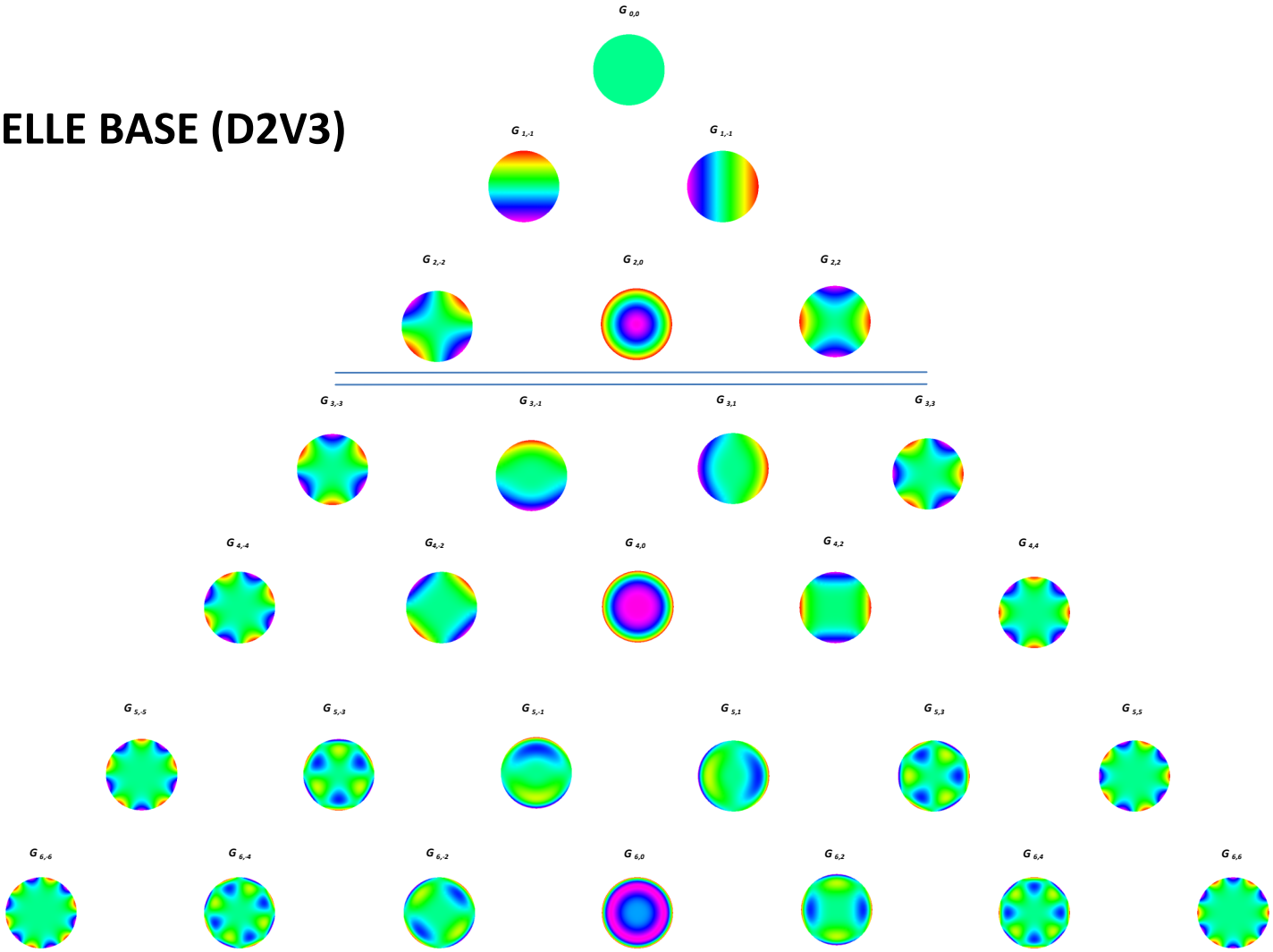
Une nouvelle base de décomposition

ANCIENNE BASE (Zernike)



Une nouvelle base de décomposition

NOUVELLE BASE (D2V3)



Une nouvelle base de décomposition

$G_{0,0}$

$$G_3^{-1}\big(r\cos(t), r\sin(t)\big) = 2\sqrt{2}\, r^3 \sin(t)$$

$$G_3^{+1}\big(r\cos(t), r\sin(t)\big) = 2\sqrt{2}\, r^3 \cos(t)$$

$$G_4^{-2}\big(r\cos(t), r\sin(t)\big) = \sqrt{10}\, r^4 \sin(2t)$$

$$G_4^{0}\big(r\cos(t), r\sin(t)\big) = \sqrt{5}\, r^4$$

$$G_4^{2}\big(r\cos(t), r\sin(t)\big) = \sqrt{10}\, r^4 \cos(2t)$$

$$G_5^{-1}\big(r\cos(t), r\sin(t)\big) = \sqrt{2}\,(5r^5 - 4r^3)\sin(t)$$

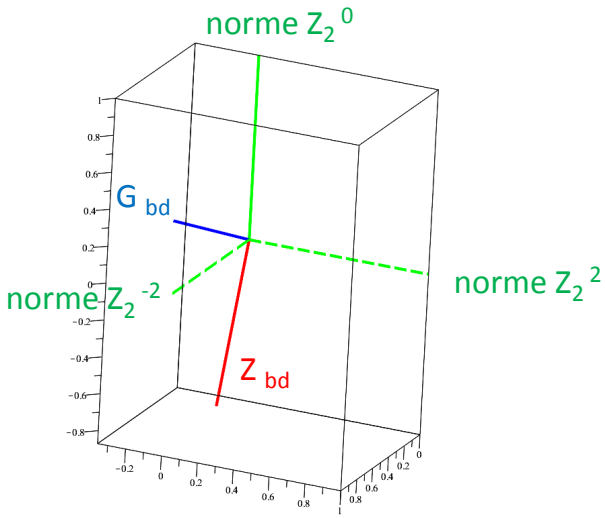
$$G_5^{+1}\big(r\cos(t), r\sin(t)\big) = 2\sqrt{2}\,(5r^5 - 4r^3)\cos(t)$$

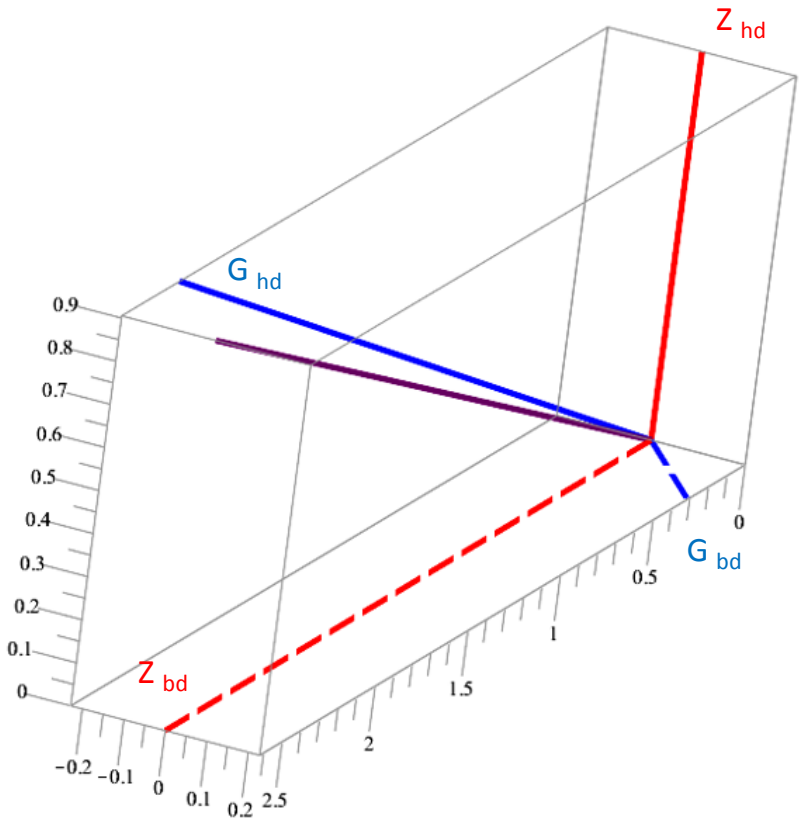
$$G_6^{-2}\big(r\cos(t), r\sin(t)\big) = \sqrt{14}\,(6r^6 - 5r^4)\sin(2t)$$

$$G_6^{0}\big(r\cos(t), r\sin(t)\big) = \sqrt{7}\,(6r^6 - 5r^4)$$

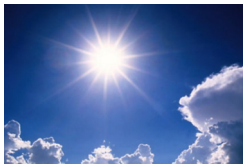
$$G_6^{2}\big(r\cos(t), r\sin(t)\big) = \sqrt{14}\,(6r^6 - 5r^4)\cos(2t)$$

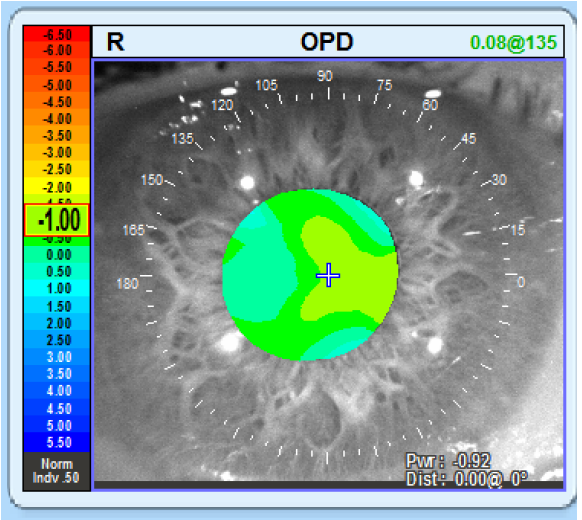
Comparaison Zernike/nouvelle base D2V3:



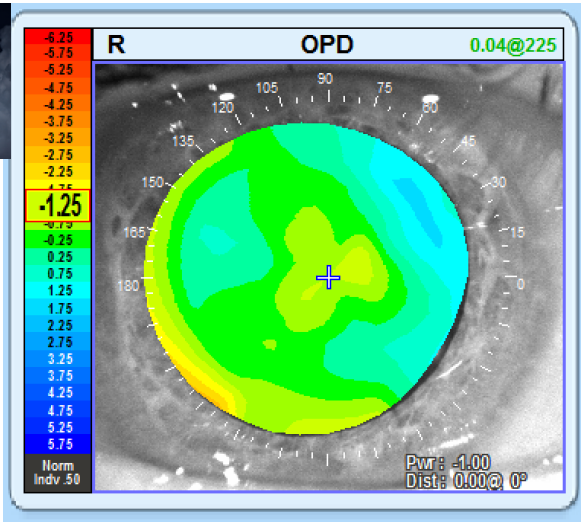


Partie 3: les résultats cliniques

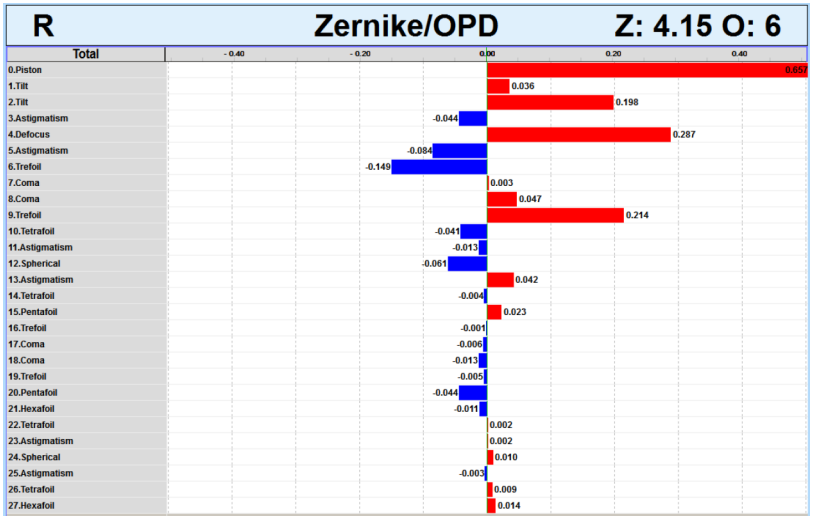


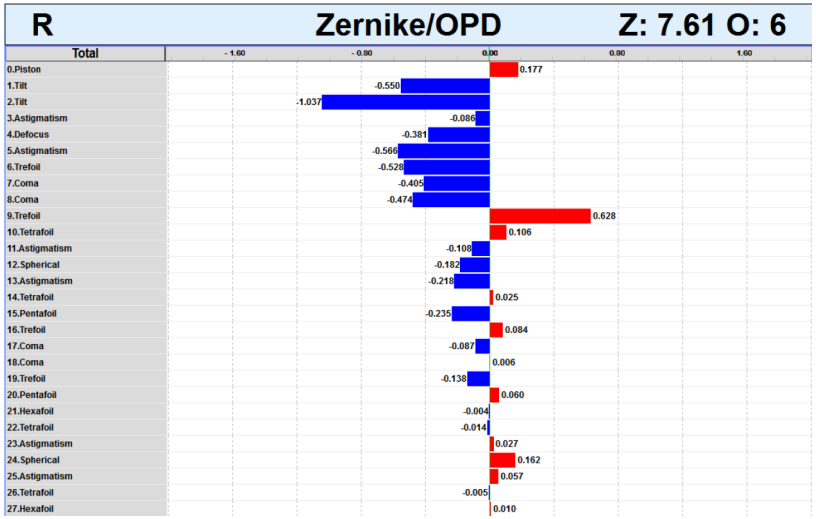


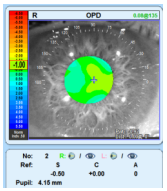
No:	2	R: /	L: /
Ref:	S	C	A
	-0.50	+0.00	0
Pupil:	4.15 mm		

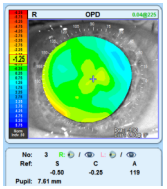


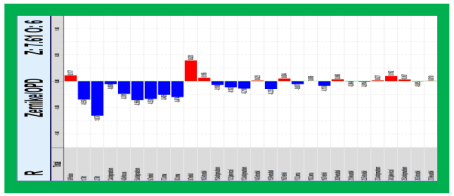
No:	3	R: /	L: /
Ref:	S	C	A
	-0.50	-0.25	119
Pupil:	7.61 mm		

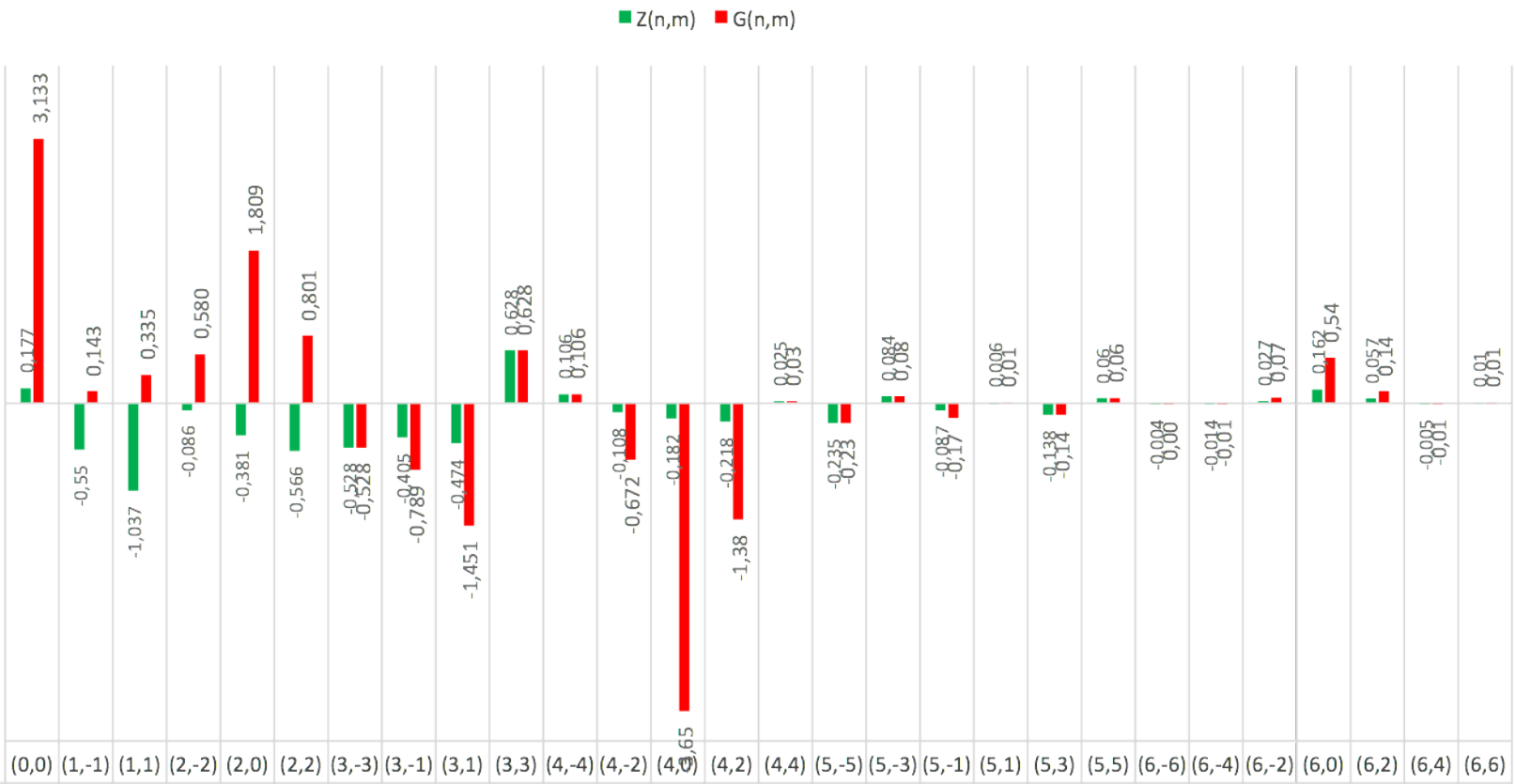






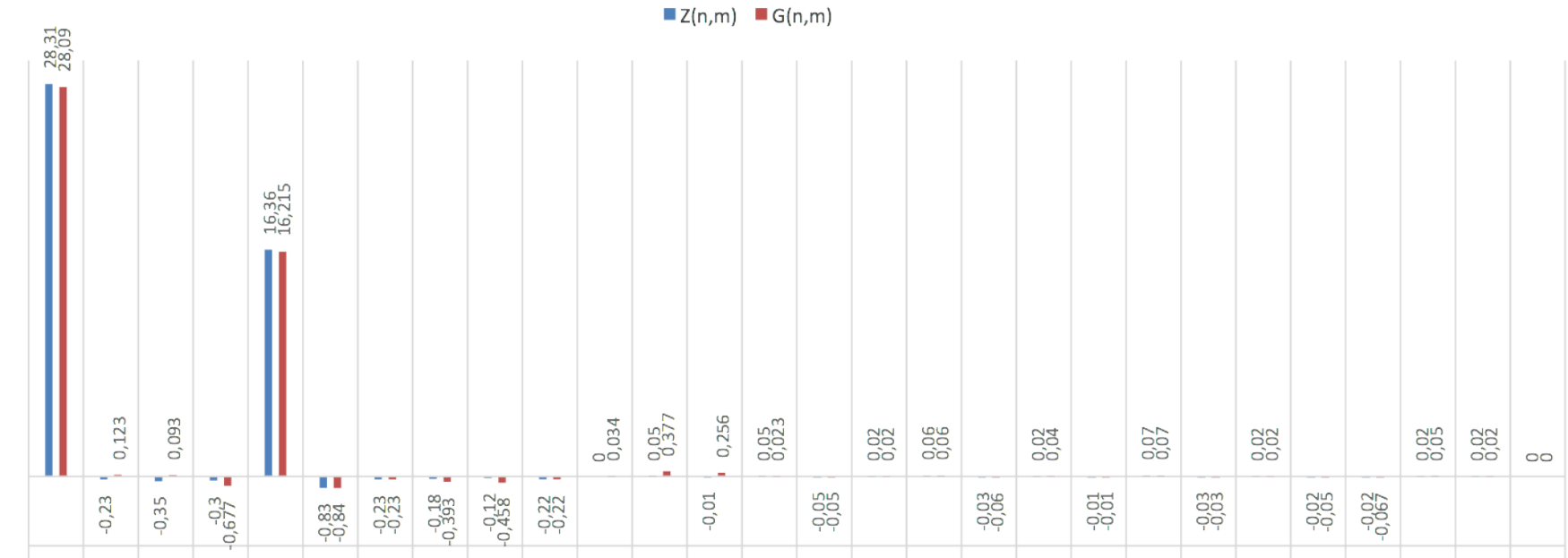






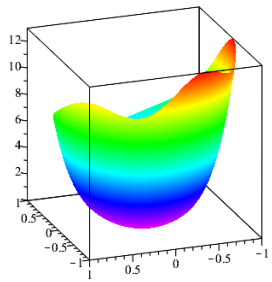
	(0,0)	(1,-1)	(1,1)	(2,-2)	(2,0)	(2,2)	(3,-3)	(3,-1)	(3,1)	(3,3)	(4,-4)	(4,-2)	(4,0)	(4,2)	(4,4)	(5,-5)	(5,-3)	(5,-1)	(5,1)	(5,3)	(5,5)	(6,-6)	(6,-4)	(6,-2)	(6,0)	(6,2)	(6,4)	(6,6)
Z(n,m)	0,177	-0,55	-1,03	-0,09	-0,38	-0,57	-0,53	-0,41	-0,47	0,628	0,106	-0,11	-0,18	-0,22	0,025	-0,24	0,084	-0,09	0,006	-0,14	0,06	-0	-0,01	0,027	0,162	0,057	-0,01	0,01
G(n,m)	3,133	0,143	0,335	0,580	1,809	0,801	-0,52	-0,78	-1,45	0,628	0,106	-0,67	-3,65	-1,38	0,03	-0,23	0,08	-0,17	0,01	-0,14	0,06	0,00	-0,01	0,07	0,54	0,14	-0,01	0,01

Comparaison Zernike/nouvelle base D2V3: myopie forte

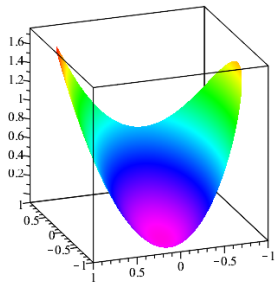


	(0,0)	(1,-1)	(1,1)	(2,-2)	(2,0)	(2,2)	(3,-3)	(3,-1)	(3,1)	(3,3)	(4,-4)	(4,-2)	(4,0)	(4,2)	(4,4)	(5,-5)	(5,-3)	(5,-1)	(5,1)	(5,3)	(5,5)	(6,-6)	(6,-4)	(6,-2)	(6,0)	(6,2)	(6,4)	(6,6)
Z(n,m)	28,31	-0,23	-0,35	-0,3	16,36	-0,83	-0,23	-0,18	-0,12	-0,22	0	0,05	-0,01	0,05	-0,05	0,02	0,06	-0,03	0,02	-0,01	0,07	-0,03	0,02	-0,02	-0,02	0,02	0,02	0
G(n,m)	28,09	0,123	0,093	-0,677	16,21	-0,84	-0,23	-0,393	-0,458	-0,22	0,034	0,377	0,256	0,023	-0,05	0,02	0,06	-0,06	0,04	-0,01	0,07	-0,03	0,02	-0,05	-0,067	0,05	0,02	0

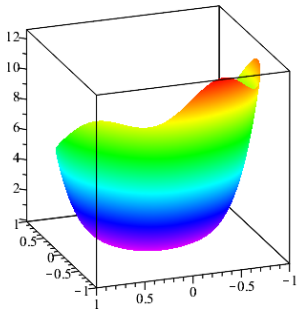
front d'onde initial



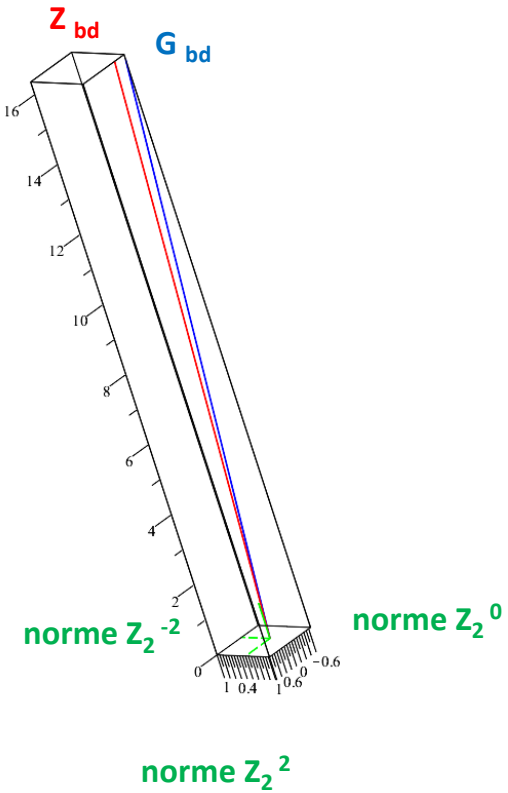
front d'onde collecté d'ordre <=2

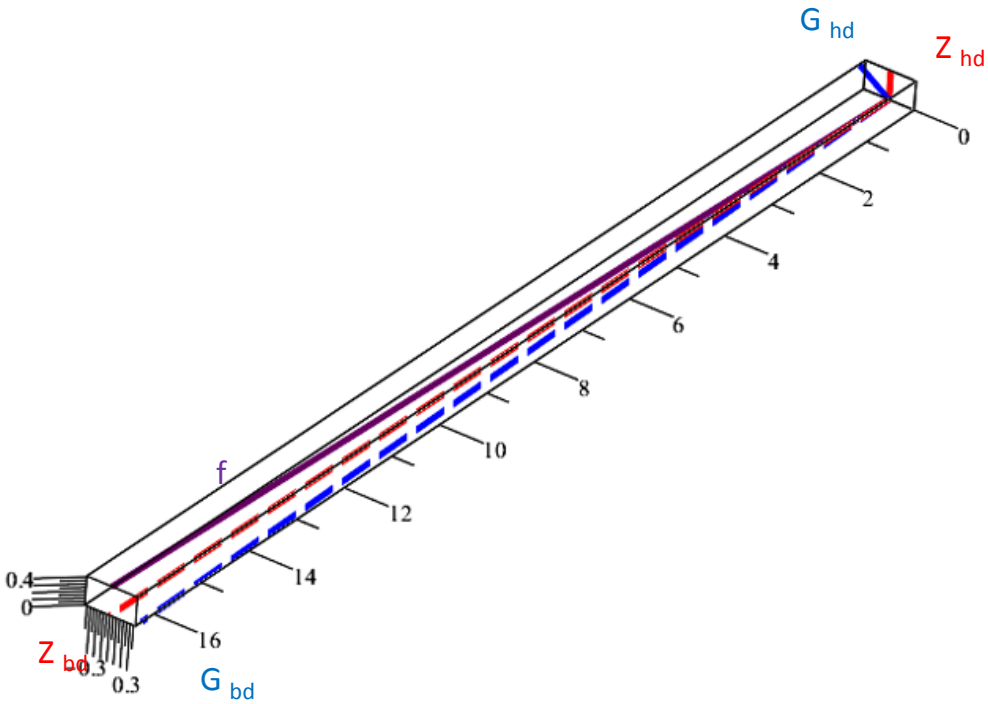


front d'onde corrigé

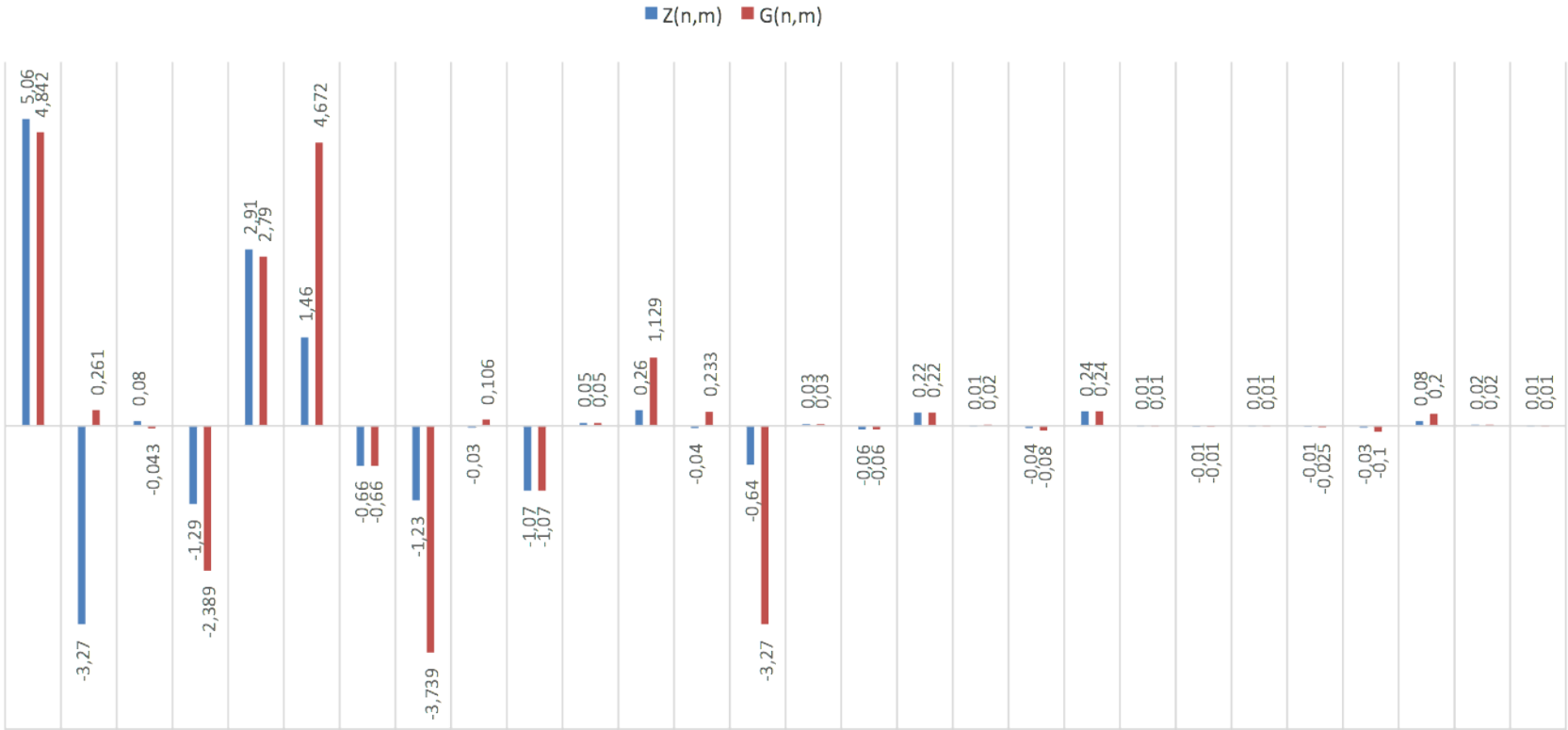


Comparaison Zernike/nouvelle base D2V3: myopie forte

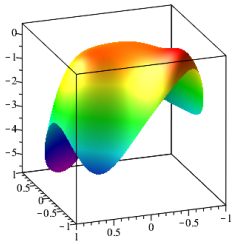


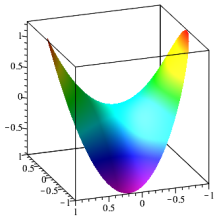


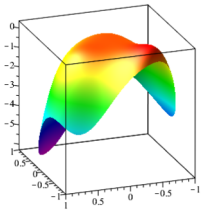
Comparaison Zernike/nouvelle base D2V3: keratocone



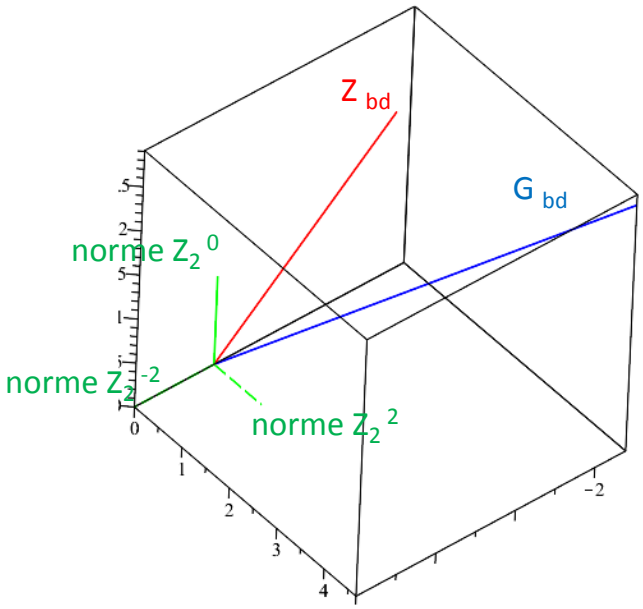
	(0,0)	(1,-1)	(1,1)	(2,-2)	(2,0)	(2,2)	(3,-3)	(3,-1)	(3,1)	(3,3)	(4,-4)	(4,-2)	(4,0)	(4,2)	(4,4)	(5,-5)	(5,-3)	(5,-1)	(5,1)	(5,3)	(5,5)	(6,-6)	(6,-4)	(6,-2)	(6,0)	(6,2)	(6,4)	(6,6)
Z(n,m)	5,06	-3,27	0,08	-1,29	2,91	1,46	-0,66	-1,23	-0,03	-1,07	0,05	0,26	-0,04	-0,64	0,03	-0,06	0,22	0,01	-0,04	0,24	0,01	-0,01	0,01	-0,01	-0,03	0,08	0,02	0,01
G(n,m)	4,842	0,261	-0,043	-2,389	2,79	4,672	-0,66	-3,739	0,106	-1,07	0,05	1,129	0,233	-3,27	0,03	-0,06	0,22	0,02	-0,08	0,24	0,01	-0,01	0,01	-0,025	-0,1	0,2	0,02	0,01

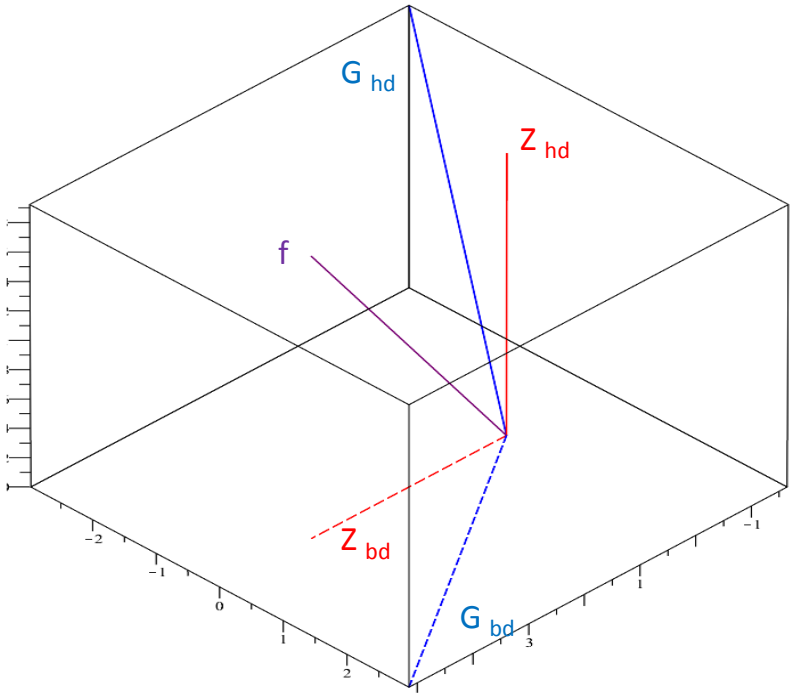




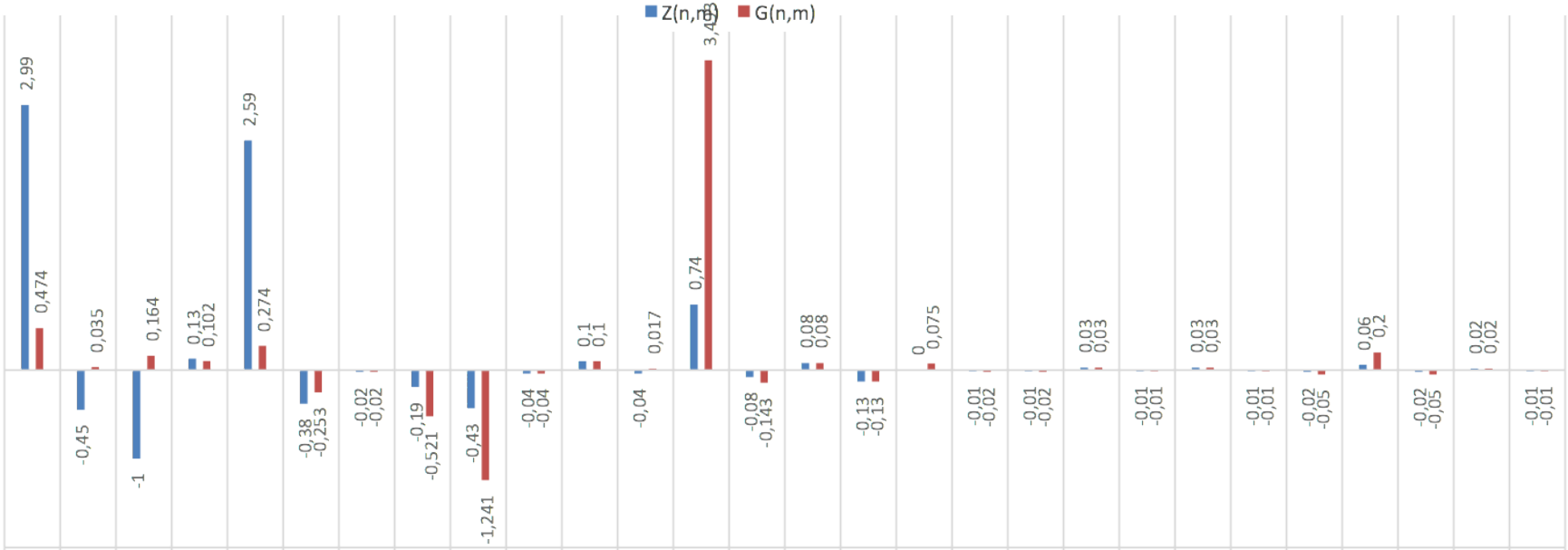


Comparaison Zernike/nouvelle base D2V3: keratocone



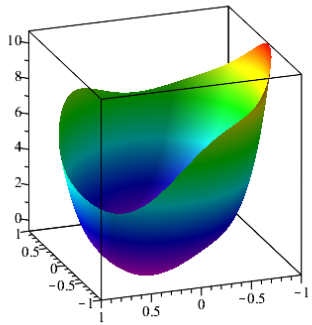


Comparaison Zernike/nouvelle base D2V3: myopie opérée

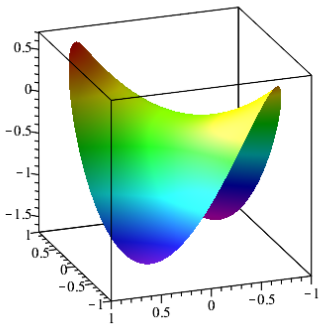


	(0,0)	(1,-1)	(1,1)	(2,-2)	(2,0)	(2,2)	(3,-3)	(3,-1)	(3,1)	(3,3)	(4,-4)	(4,-2)	(4,0)	(4,2)	(4,4)	(5,-5)	(5,-3)	(5,-1)	(5,1)	(5,3)	(5,5)	(6,-6)	(6,-4)	(6,-2)	(6,0)	(6,2)	(6,4)	(6,6)
Z(n,m)	2,99	-0,45	-1	0,13	2,59	-0,38	-0,02	-0,19	-0,43	-0,04	0,1	-0,04	0,74	-0,08	0,08	-0,13	0	-0,01	-0,01	0,03	-0,01	0,03	-0,01	-0,02	0,06	-0,02	0,02	-0,01
G(n,m)	0,474	0,035	0,164	0,102	0,274	-0,253	-0,02	-0,521	-1,241	-0,04	0,1	0,017	3,493	-0,143	0,08	-0,13	0,075	-0,02	-0,02	0,03	-0,01	0,03	-0,01	-0,05	0,2	-0,05	0,02	-0,01

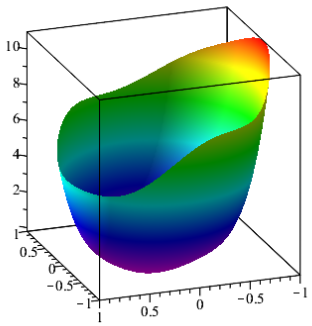
front d'onde initial



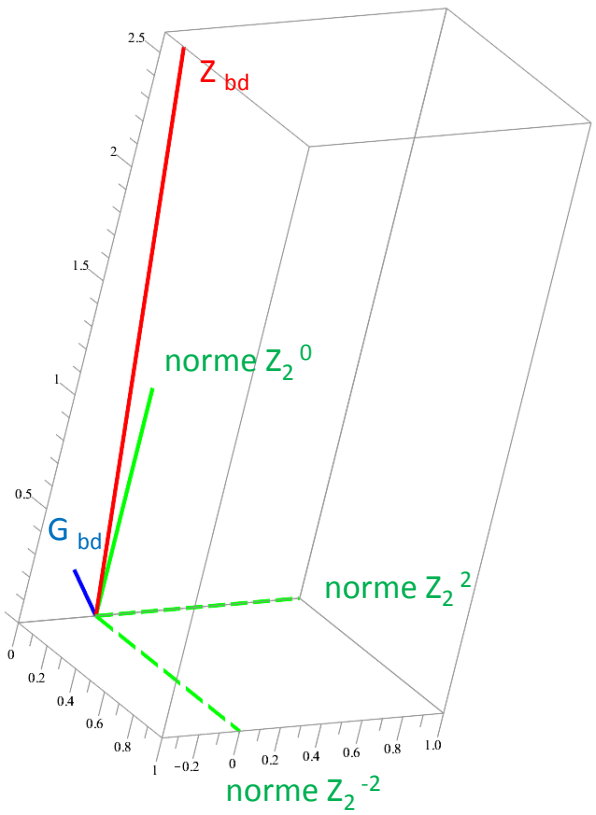
front d'onde collecté d'ordre <=2

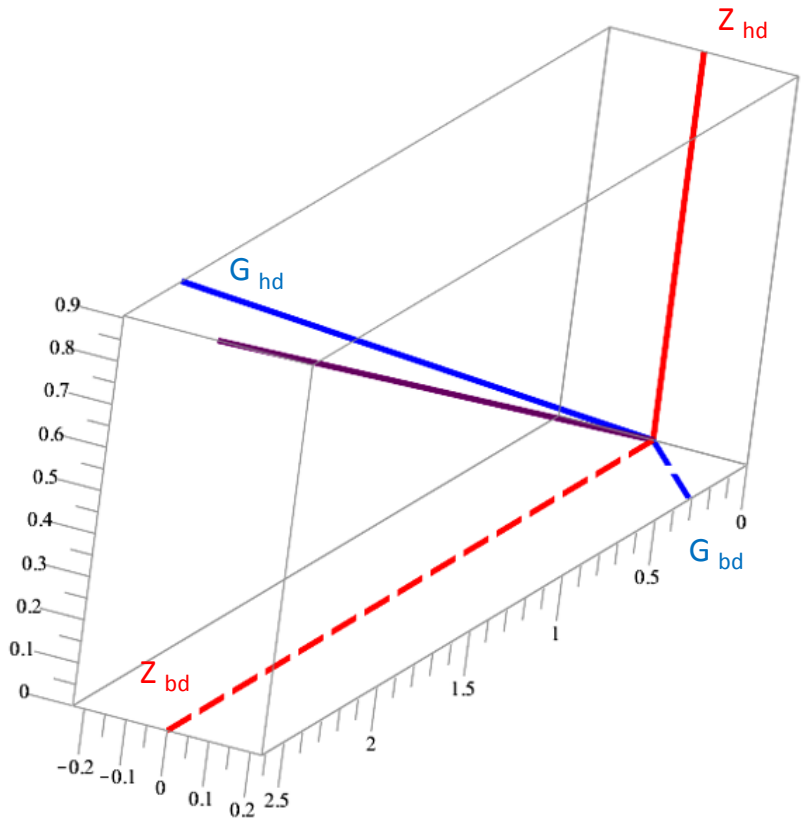


front d'onde corrigé

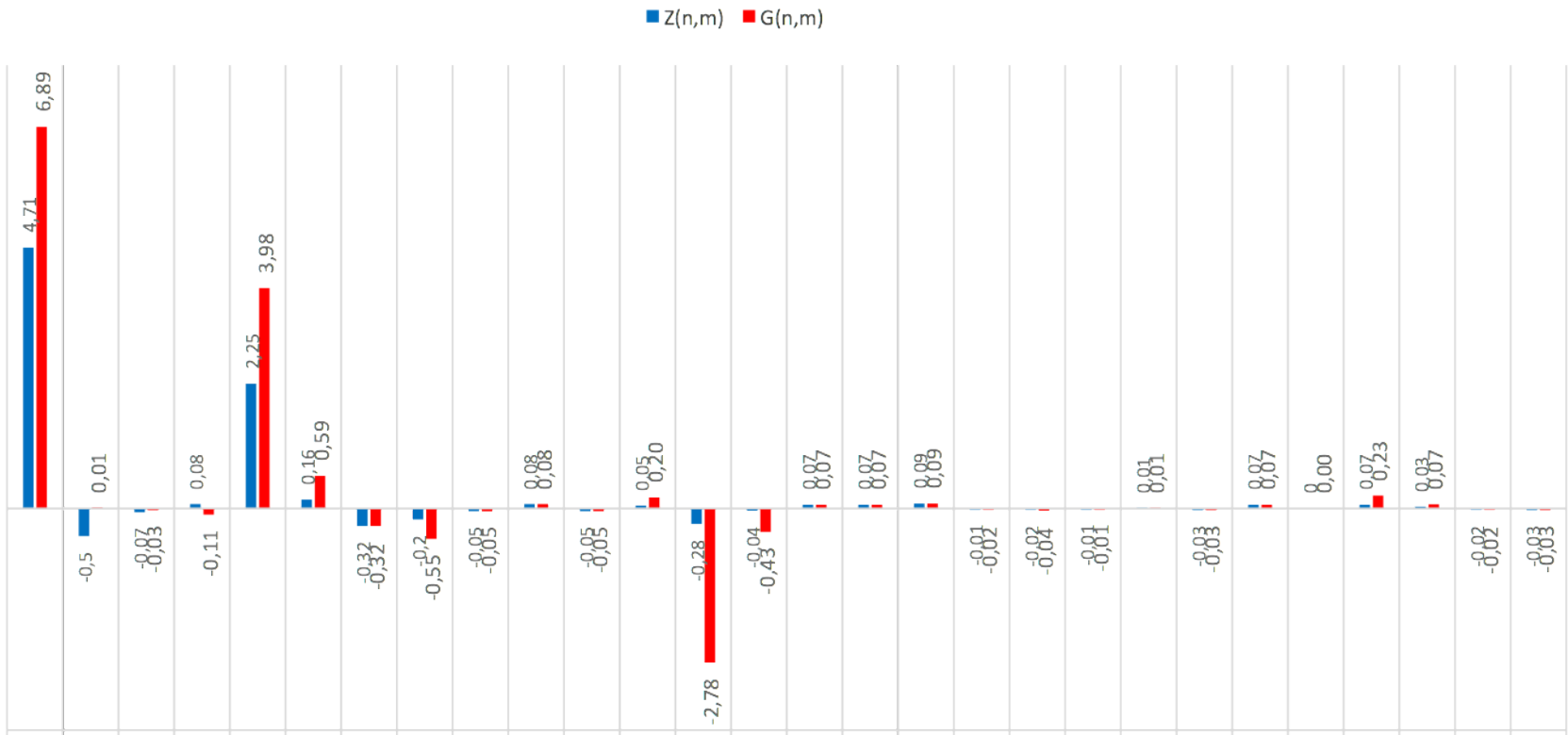


Comparaison Zernike/nouvelle base D2V3: myopie opérée



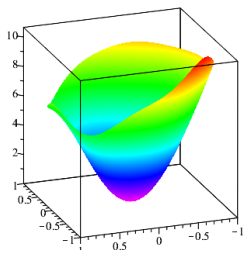


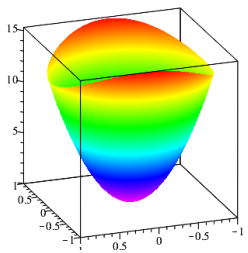
Comparaison Zernike/nouvelle base D2V3: cataracte nucléaire

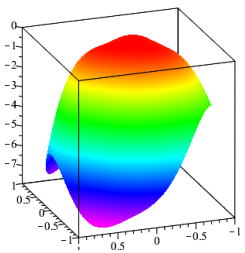


	(0,0)	(1,-1)	(1,1)	(2,-2)	(2,0)	(2,2)	(3,-3)	(3,-1)	(3,1)	(3,3)	(4,-4)	(4,-2)	(4,0)	(4,2)	(4,4)	(5,-5)	(5,-3)	(5,-1)	(5,1)	(5,3)	(5,5)	(6,-6)	(6,-4)	(6,-2)	(6,0)	(6,2)	(6,4)	(6,6)
Z(n,m)	4,71	-0,5	-0,07	0,08	2,25	0,16	-0,32	-0,2	-0,05	0,08	-0,05	0,05	-0,28	-0,04	0,07	0,07	0,09	-0,01	-0,02	-0,01	0,01	-0,03	0,07	0	0,07	0,03	-0,02	-0,03
G(n,m)	6,89	0,01	-0,03	-0,11	3,98	0,59	-0,32	-0,55	-0,05	0,08	-0,05	0,20	-2,78	-0,43	0,07	0,07	0,09	-0,02	-0,04	-0,01	0,01	-0,03	0,07	0,00	0,23	0,07	-0,02	-0,03

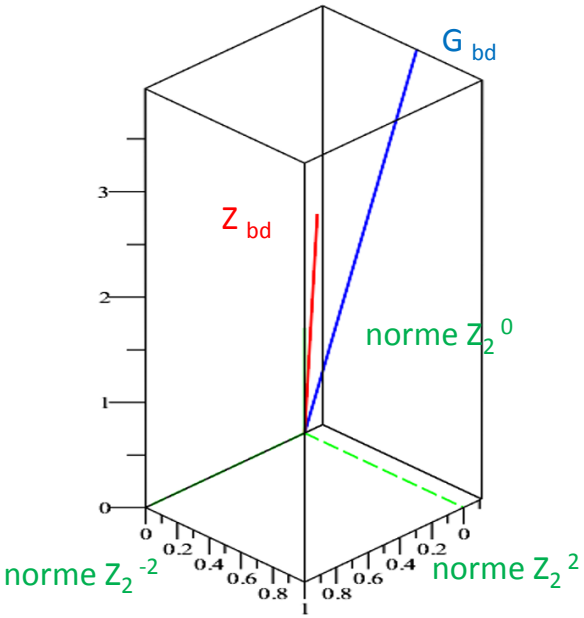
front d'onde initial *front d'onde collecté d'ordre <=2* *front d'onde collecté d'ordre >=3*

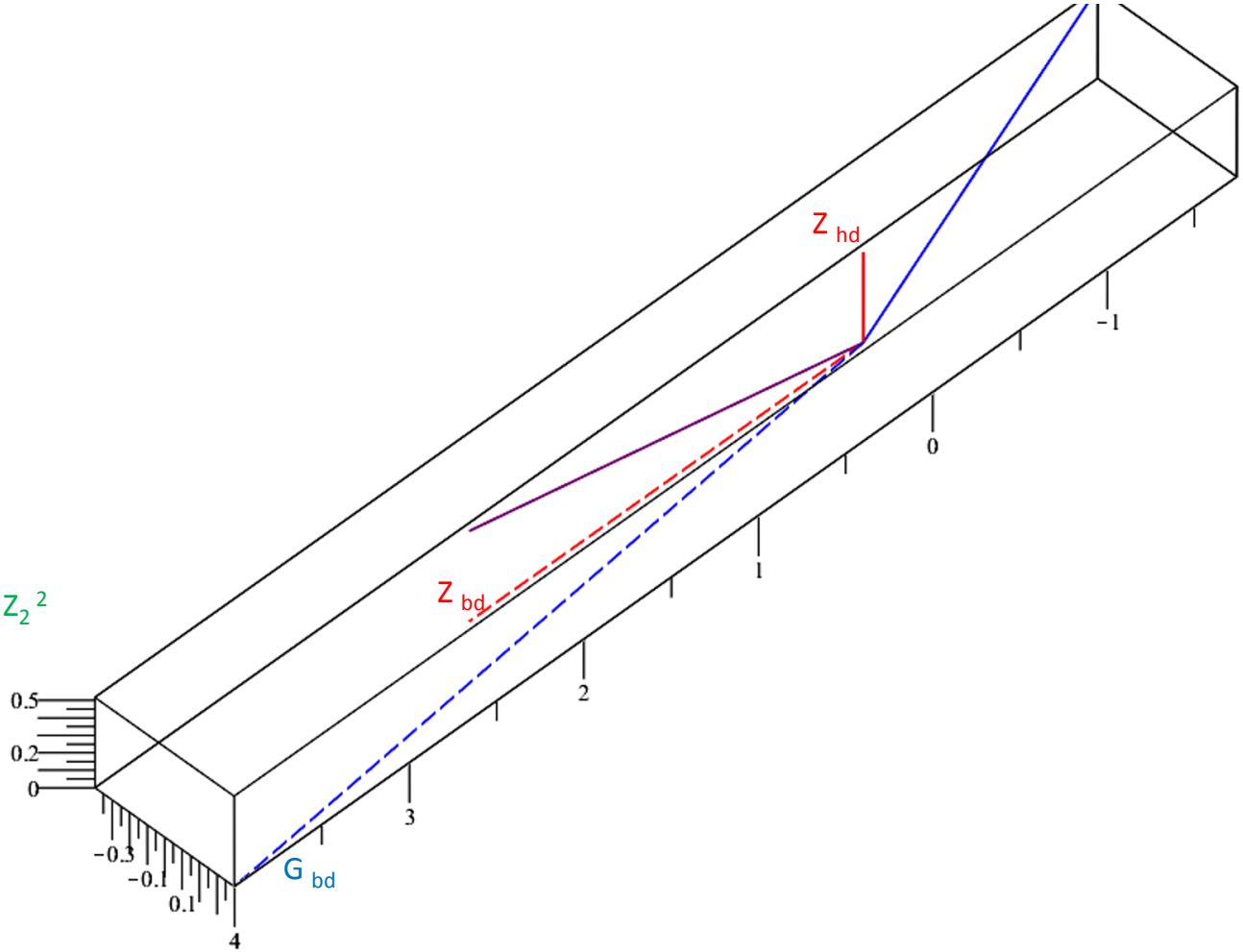






Comparaison Zernike/nouvelle base D2V3: cataracte nucléaire





Conclusion et problèmes en cours

Conclusion et problèmes en cours

Sur l'exemple présenté, les outils mathématiques développés ont pu améliorer le diagnostic médical des défauts de vision.

De nombreux autres problèmes en ophtalmologie sont encore en cours de modélisation et/ou simulation, liés en particulier à l'opération de la cornée par ablation au laser.

MERCI POUR VOTRE ATTENTION

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