Age-structured model in subdiffusion

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Model

$u(t,x,a)$: density of particles of age $a \in \mathbb{R}_+$ at time $t \in \mathbb{R}_+$ and position $x \in \mathbb{R}_+^n$.

Age renewal equation with jumps that restart the age at 0 :

$$\begin{cases} \partial_t u(t,x,a) + \partial_a u(t,x,a) + \beta(a)u(t,x,a) = 0 \\ u(t,x,0) = \int_0^\infty \int_{\mathbb{R}} \beta(a) w(x-x') u(t,x',a) \mathrm{d}x \mathrm{d}a \\ u(0,x,a) = u^0(x,a). \end{cases} \tag{1}$$

β : jump frequency.

w : space distribution of jumps.

Our secret goal: $\beta = \beta(a,x)$.

Pure age structured model

We consider the dynamics of age . We integrate everything into space.

$$\begin{cases} \partial_t n(t,a) + \partial_a n(t,a) + \beta(a)n(t,a) = 0 \\ n(t,0) = \int_0^\infty \beta(a)n(t,a)\mathrm{d}x\mathrm{d}a \\ n(0,a) = n^0(a). \end{cases}$$

The problem is linear. Without loss of generality we will always assume $n^0 \geq 0$ (to make sense) and

$$\int n^0 = 1.$$

Interpretations of β

$\beta \geq 0$ is the rate of jump (renewal).

$$P(\text{no jump before reaching age a}) = e^{-\int_0^a \beta(a')da'}$$

Assume $\int_0^\infty \beta = +\infty$, then 'everyone eventually jumps'.

$$\beta(a)e^{-\int_0^a \beta(a')da'}da$$

is the probability distribution of the jumps.

$$\int_0^\infty a\beta(a)e^{-\int_0^a \beta(a')da'}da$$

is the mean waiting time between two jumps. It CAN BE $+\infty$!

Classical case

Consider β bounded and $\beta(a) \geq \beta_0 > 0$, then there is a unique steady state

$$\begin{cases} \partial_a N(a) + \beta(a) N(a) = 0 \\ N(0) = \int_0^\infty \beta(a) N(a) \mathrm{d}x\mathrm{d}a. \end{cases}$$

It can be written as $N(a) = N(0) e^{-\int_0^a \beta(a') da'}$ and we have

$$\int_0^\infty N(a) da = 1.$$

Convergence to steady state

Since $\partial_t N = 0$, we can also write $\tilde{n} = n - N$ and $\tilde{n}$ satisfies

$$\begin{cases} \partial_t \tilde{n}(t,a) + \partial_a \tilde{n}(t,a) + \beta(a)\tilde{n}(t,a) = 0 \\ \tilde{n}(t,0) = \int_0^\infty \beta(a)\tilde{n}(t,a)\mathrm{d}x\mathrm{d}a \\ \tilde{n}(0,a) = n^0(a) - N(a). \end{cases}$$

We multiply the equation by $sign(n(t,a))$ and get, since

$$\partial_t |\tilde{n}(t,a)| + \partial_a |\tilde{n}(t,a)| + \beta(a)|\tilde{n}(t,a)| = 0$$

We integrate

$$\frac{d}{dt}\int |\tilde{n}| = -\int_0^\infty \beta(a)|\tilde{n}| + |\tilde{n}(t,0)|$$
$$= -\int_0^\infty \beta(a)|\tilde{n}| + \left|\int_0^\infty \beta(a)\tilde{n}\right|$$

Now a small trick: since $\beta \geq \beta_0 > 0$ and $\int \tilde{n} = 1 - 1 = 0$,

$$\frac{d}{dt}\int |\tilde{n}| = -\beta_0 \int_0^\infty |\tilde{n}| \underbrace{- \int_0^\infty (\beta(a)-\beta_0)|\tilde{n}| + \left|\int_0^\infty (\beta(a)-\beta_0)\tilde{n}\right|}_{\leq 0 \quad \text{(triangle inequality)}}$$

Finally,

$$\int^\infty |\tilde{n} - N|(t,a) < Ce^{-\beta_0 t} \to 0.$$

Other scenarios: trying to guess

If $\int_0^\infty e^{-\int_0^a \beta} da < +\infty$ we can define the steady state $N(a)$ (and prove convergence of solutions to N, at least without a rate). What happens in other cases? Let us have a look at some moments. Firstly, formally, for φ smooth

$$\frac{d}{dt}\int_0^\infty \varphi(a)n(t,a)da = \varphi(0)n(t,0) + \int_0^\infty \varphi'(a)n(t,a)da - \int_0^\infty \beta(a)\varphi(a)n(t,a)da$$

Choice 1 (exercise): $\varphi(a) = e^{\int_0^a \beta} \int_0^a e^{-\int_0^{a'} \beta} da'$

From now on,

$$\beta(a) = \frac{\mu}{1+a}, \quad 0 < \mu < 1$$

In this case, we have both

$$\int_0^\infty e^{-\int \beta} = +\infty, \quad \int a\beta(a) e^{-\int_0^a \beta} da = +\infty.$$

The individuals jump but not so often (mean waiting time is infinite).
Moreover we will assume that we start with young individuals $suppn^0 \subset [0,1[$.

First step to scaling

Compute

$$\frac{d}{dt}\int_0^\infty an(t,a)da = \int_0^\infty (1-\frac{\mu a}{1+a})n(t,a)da \geq (1-\mu)\int n = (1-\mu).$$

Mean age of the population grows satisfy

$$(1-\mu) \leq \frac{d}{dt}\int an \leq 1$$

Mean age of the population is expected to grow linearly! We do not expect $n \to N$! Next idea: do we have

$$n \sim \frac{1}{t} f(a/t)?$$

Scaling

We choose another point of view: zoom out and observe a/t.
Write

$$n(t,a) = \frac{1}{(1+t)} w\left(\tau(t), \frac{a}{1+t}\right).$$

Why change as well time scale? Rewrite the equation

Denote the variables $w(\tau, b)$, at $\tau, b = \tau(t), \frac{a}{1+t}$,

$$\partial_t n = -\frac{1}{(1+t)^2} w + \frac{\tau'(t)}{1+t} \partial_\tau w - \frac{1}{(1+t)} \frac{a}{1+t} \partial_b w$$

Homogeneity argument to choose τ:

$$\partial_t n = -\frac{1}{(1+t)^2} \left(w + \tau'(t)(1+t) \partial_\tau w - b \partial_b w \right)_{(\tau,b)=(\tau(t), \frac{a}{1+t}}$$

Choose $\tau' = \frac{1}{1+t}$, i.e $\tau(t) = \log(1+t)$ (and $1 + t = e^\tau$) and get

$$\partial_t n = \frac{1}{(1+t)^2} \left(\partial_\tau w - w - b \partial_b w \right)_{(\tau,b)=(\tau(t), \frac{a}{1+t}}$$

Other terms

$$\partial_a n = \frac{1}{(1+t)^2} \partial_b w|_{\tau, b=\tau(t), a/(1+t)}$$

Transport part gives

$$\begin{aligned}
\partial_t n + \partial_a n &= \frac{1}{(1+t)^2} \left(\partial_\tau w + (1-b)\partial_b w - w\right)_{(\tau,b)=(\tau(t),\frac{a}{1+t})} \\
&= \frac{1}{(1+t)^2} \left(\partial_\tau w + \partial_b((1-b)w)\right)_{(\tau,b)=(\tau(t),\frac{a}{1+t})}
\end{aligned}$$

$$\begin{aligned}
\frac{\mu}{1+a} n(t,a) &= \frac{\mu}{1+a} \frac{1}{1+t} w(\tau, b)_{(\tau,b)=(\tau(t),\frac{a}{1+t})} \\
&= \frac{1}{(1+t)^2} \left(\frac{\mu}{e^{-\tau}+b} w(\tau, b) \right)
\end{aligned}$$

Finally

$$\partial_t n+\partial_a n+\frac{\mu}{1+a}n(t,a)=\frac{1}{(1+t)^2}\left(\partial_\tau w+\partial_b((1-b)w)+\frac{\mu}{e^{-\tau}+b}\right)$$

With this scaling, w satisifies

$$\begin{cases}\partial_\tau w+\partial_b((1-b)w)+\frac{\mu}{e^{-\tau}+b}w=0,\\ w(\tau,0)=\int_0^\infty \frac{\mu}{e^{-\tau}+b}w(\tau,b)db.\end{cases} \tag{2}$$

Note that the important equality

$$\int_0^\infty \left| n(t,a)-\frac{1}{1+t}\bar{w}\left(\frac{a}{1+t}\right)\right| da=\int_0^\infty |w(\tau(t),b)-\bar{w}(b)|db$$

A few remarks

The equation is not autonomous and does not admit a steady state. If we start with a support in $[0,1[$, this keeps true. Educated guess: look for solutions of the equations for $\tau = +\infty$ with support in $[0,1]$.

$$\partial_b((1-b)\bar{w}) + \frac{\mu}{b}\bar{w} = 0$$

Solution

$$\bar{w} = \frac{C}{b^{\mu}(1-b)^{1-\mu}}$$

Problem:

$$\bar{w}(0) = +\infty, \text{ BUT ALSO } \int_0^1 \frac{\mu}{b}\bar{w}(b) = +\infty$$

We can not inject $\bar{w}$ in the equation. Idea, find a ansatz $W(\tau, b)$ which

- converges to $\bar{w}$ for large time
- satisifies almost the same equation than w.

Natural[1] candidate

$$W(\tau, b) = \frac{C(\tau)}{(e^{-\tau} + b)^{\mu}(1 - b)^{1-\mu}}$$

We call it pseudo equilibrium.

[1] only a few months to identify it!

Algebraic miracle

$$\partial_\tau W + \partial_b((1-b)W) + \frac{\mu}{e^{-\tau}+b}W = \frac{C'(\tau)}{C(\tau)}W$$

For the boundary condition, we have

$$W(\tau,0) = \begin{cases} C(\tau)e^{\mu\tau} \\ (1+e^{-\tau})\int_0^1 \frac{\mu}{e^{-\tau}+b}W(\tau,b) \quad \text{(direct computations)}, \\ \int_0^1 \frac{\mu}{e^{-\tau}+b}W(\tau,b) - \frac{C'}{C} \quad \text{(using the fact that } \int_0^1 W = 1\text{)}. \end{cases}$$

W is closed to $\bar{w}$

Preliminary but important remark.

$$\frac{1}{C} = \int_0^1 \frac{1}{(e^{-\tau}+b)^{\mu}(1-b)^{1-\mu}}$$

$$\begin{aligned} -\frac{C'}{C^2} &= \int_0^1 \frac{\mu e^{-\tau}}{(e^{-\tau}+b)^{\mu+1}(1-b)^{1-\mu}} \\ &= \frac{e^{-\tau}}{1+e^{-\tau}}\left[-(1-b)^{\mu}(e^{-\tau}+b)^{-\mu}\right]_0^1 \\ &= \frac{e^{(\mu-1)\tau}}{1+e^{-\tau}} \end{aligned}$$

In particular, since C is bounded from below and above, we will have

$$\left|\frac{C'}{C}\right|, |C'|, |C - C_\infty| \leq K e^{(\mu-1)\tau}.$$

$$\int |W - \bar{w}| = \int \frac{1}{(1-b)^{1-\mu}} \left| \frac{C(\tau)}{(e^{-\tau}+b)^{\mu}} - \frac{C_\infty}{b^{\mu}} \right|$$
$$\leq \int \frac{1}{(1-b)^{1-\mu}} \left| \frac{C_\infty}{(e^{-\tau}+b)^{\mu}} - \frac{C_\infty}{b^{\mu}} \right|$$
$$+ \underbrace{\int \frac{1}{(1-b)^{1-\mu}} \left| \frac{C(\tau)-C_\infty}{(e^{-\tau}+b)^{\mu}} \right|}_{\leq K e^{(\mu-1)\tau}}$$

We need to estimate

$$\int \frac{1}{(1-b)^{1-\mu}} \left| \frac{1}{(e^{-\tau}+b)^{\mu}} - \frac{1}{b^{\mu}} \right| = \int_0^1 \frac{1}{b^{\mu}(1-b)^{1-\mu}} \left| \frac{b^{\mu}}{(e^{-\tau}+b)^{\mu}} - 1 \right|$$

$$= \int_0^1 \frac{1}{b^{\mu}(1-b)^{1-\mu}} \int_{\tau}^{\infty} \frac{d}{d\tau} \frac{b^{\mu}}{(e^{-\tau}+b)^{\mu}}$$

$$= \int_0^1 \int_{\tau}^{\infty} \frac{\mu e^{-\tau}}{(e^{-\tau}+b)^{\mu+1}(1-b)^{1-\mu}}$$

Interverting the integrals we have

$$\int \frac{1}{(1-b)^{1-\mu}} \left| \frac{1}{(e^{-\tau}+b)^{\mu}} - \frac{1}{b^{\mu}} \right| = \frac{e^{(\mu-1)\tau}}{1+e^{-\tau}} \leq K e^{(\mu-1)\tau}$$

Conclusion

$$\int_0^1 |W - \bar{w}| \leq K e^{(\mu-1)\tau}.$$

w becomes close to W.

Denoting $s(\tau, b) = sign(w - W)$, we have

$$\partial_\tau |w-W| + \partial_b((1-b)|w-W|) + \frac{\mu}{e^{-\tau}+b}|w-W| = -\frac{C'(\tau)}{C(\tau)} W s(\tau, b)$$

Integrating, we get

$$\begin{aligned}\frac{d}{d\tau}\int_0^1 |w-W| = &-\int_0^1 \frac{\mu}{e^{-\tau}+b}|w-W| \\ &-\frac{C'}{C}\int_0^1 Ws \\ &+\left|\int_0^1 \frac{\mu}{e^{-\tau}+b}(w-W)+\frac{C'}{C}\right|\end{aligned}$$

We denote $I = \int |w - W|$. Using triangle inequality we have easily

$$I' \leq 2\left|\frac{C'}{C}\right| + \left|\int_0^1 \frac{\mu}{e^{-\tau}+b}(w-W)\right| - \int_0^1 \frac{\mu}{e^{-\tau}+b}|w-W|$$

Same trick as before $\int w - W = 0$, $\frac{\mu}{e^{-\tau}+b} - \frac{\mu}{e^{-\tau}+1} \geq 0$

$$\begin{aligned} I' &\leq 2\left|\frac{C'}{C}\right| - \frac{\mu}{1+e^{-\tau}}\int_0^1 |w-W| \\ &\leq 2Ke^{(\mu-1)\tau} - \frac{\mu}{1+e^{-\tau}}\int_0^1 |w-W| \\ &\leq 2Ke^{(\mu-1)\tau} - \mu I + \frac{\mu e^{-\tau}}{1+e^{-\tau}}\int_0^1 |w-W| \\ &\leq (2K+2\mu)e^{(\mu-1)\tau} - \mu I \end{aligned}$$

We have then

$$(Ie^{\mu\tau})' \leq Ae^{(2\mu-1)\tau}$$

And thereby

$$\begin{aligned} I(\tau) &\leq I(0)e^{-\mu\tau} + Ae^{-\mu\tau}\int_0^\tau e^{(2\mu-1)\tau'}d\tau' \\ &\leq I(0)e^{-\mu\tau} + \begin{cases} Be^{(\mu-1)\tau} \text{ if } \mu \neq 1/2, \\ B\tau e^{-\tau/2} \text{ if } \mu = 1/2. \end{cases} \end{aligned}$$

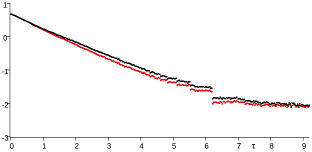
Putting things together

We have estimated the two interesting quantities. We end up with the estimates

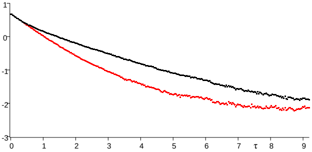
$$\int_0^1 |w - \bar{w}| \leq C_1(\mu)e^{-\mu\tau} + C_2(\mu)e^{-(1-\mu)\tau} + \chi_{\mu=\frac{1}{2}} C_{\frac{1}{2}} \tau e^{-\tau/2}.$$

Not only this converges but the convergence is exponential! The rate is $e^{-\min(\mu,1-\mu)\tau}$ (with a correction in case $\mu = 1/2$).

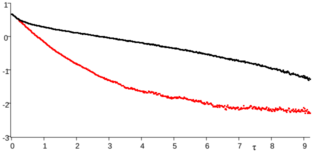
Was it a good idea to introduce W?



$\mu = 0.5$



$\mu = 0.7$



$\mu = 0.9$

Figure: Influence of μ on $\ln \|w(\tau,\cdot) - W(\tau,\cdot)\|_1$ (red dots) and $\ln \|w - W_\infty\|$ (black dots): for higher values of μ, w is significantly closer to W than to W_∞.

Possible improvements

If $\beta = \frac{\mu}{1+a} + o\left(\frac{1}{1+a}\right)$, we can also give a rate BUT it greatly depends of what is $o\left(\frac{1}{1+a}\right)$

Next steps

A step towards space dependency. Multi compartments

$$\begin{cases} \partial_t n_i + \partial_a n_i + \frac{\mu_i}{1+a} n_i = 0, \\ n_i(t,0) = \int_0^\infty \sum_j \frac{m_{ij}\mu_j}{1+a} n_j da. \end{cases}$$

Where M is a stochastic matrix $m_{ij} \geq 0$, and $\sum_j m_{ij} = 1$ for all i.
Pseudo equilibrium-$>$ done
convergence is much harder (even without a rate)
Large deviation approach (not me: H. Berry, V. Calvez P. Gabriel[2], A. Mateos Gonzales).

[2] Yes, the same one

Thanks

- H. Berry and A. Mateos Gonzales
  https://arxiv.org/abs/1503.08552
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