Recovering the History of a Fibril Population Undergoing Fragmentation from Its Asymptotic Profile

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The generic problem

Goal

Understand the dynamical behaviour of a system of fibrils undergoing a mechanical fragmentation.

Context: Diseases involving Amyloid fibrils (Prion, Alzheimer).

Tools: Deterministic models of the fragmentation mechanism.

Data: Experiment performed at the University of Kent, UK, by the team of Prof Xue.
1. The fragmentation equation

2. Motivation: A new experiment

3. Uniqueness and for the inverse problem

4. Numerical simulations and application to real data
The fragmentation equation

Motivation: A new experiment

Uniqueness and for the inverse problem

Numerical simulations and application to real data
A fragmentation equation

Deterministic equation describing the evolution of a population of particles with characteristic size $x \sim \text{Size-structured PDE.}$

**Unknown:** $f(t,x) = \text{Density of particles of size } x \text{ at time } t.$

\[
\frac{\partial f}{\partial t}(t, x) = -B(x)f(t, x) + \int_{y=x}^{y=\infty} k(x, y)B(y)f(t, y)dy
\]

**Parameters:**

- $B(x) = \text{Fragmentation rate of particles of size } x.$
  Classical assumption: $B(x) = \alpha x^\gamma, \gamma > 0, \alpha > 0,$

- $k(x, y) = \text{Fragmentation kernel}.$
  Classical assumptions: $k(x, y) = \frac{1}{y}k_0\left(\frac{x}{y}\right),$ where $k_0$ is a measure on $[0, 1],$

  \[
supp(k_0) \subset [0, 1], \quad \int_0^1 dk_0(z) = 2, \quad \int_0^1 zdk_0(z) = 1.
\]
Related size-structured PDE

- **Pure Fragmentation.** Particles can only break up.
  Example: Amyloid fibrils
  \[
  \frac{\partial f}{\partial t}(t, x) = -B(x)f(t, x) + \int_{y=x}^{y=\infty} k(x, y)B(y)f(t, y)dy
  \]

- **Transport-Fragmentation.** Particles can break up and grow.
  Example: Cells, microtubules
  \[
  \frac{\partial f}{\partial t}(t, x) + \frac{\partial}{\partial x} (g(x)f(t, x)) = -B(x)f(t, x) + \int_{y=x}^{y=\infty} k(x, y)B(y)f(t, y)dy
  \]

- **Coagulation-Fragmentation.** Particles can break up or merge.
  Example: Dispersion of dust, smoke, or pollutants.
  \[
  \frac{\partial f}{\partial t}(t, x) = -B(x)f(t, x) + \int_{y=x}^{y=\infty} k(x, y)B(y)f(t, y)dy
  \]
  \[
  -f(t, x) \int_{0}^{+\infty} k_c(x, y)f(y)dy + \int_{y=0}^{y=x} k_c(y, x-y)f(t, y)f(t, x-y)dy
  \]
Existence and basic properties

The fragmentation equation:

\[
\begin{aligned}
&\frac{\partial f}{\partial t}(t, x) = -B(x)f(t, x) + \int_{y=x}^{y=\infty} \frac{B(y)}{y} k_0 \left( \frac{x}{y} \right) f(t, y) dy, \\
&f(x, 0) = f_0(x).
\end{aligned}
\]

(1)

Under the assumptions on the fragmentation rate \((B(x) = x^\gamma)\) and on the scaling of the kernel, the fragmentation equation has a global solution (Smith, Thieme, Escobedo, Michel, Perthame, Mishler, ...) which lies in \(C([0, T); L^1(\mathbb{R}^+, xdx)) \cap L^1(0, T; L^1(\mathbb{R}^+, x^{\gamma+1} dx)).\)

Conservation properties:

\[
\frac{d}{dt} \int_0^{+\infty} f(t, x) dx = \int_0^{+\infty} B(x)f(t, x) dx \quad \text{Number of clusters increases}
\]

\[
\frac{d}{dt} \int_0^{+\infty} xf(t, x) dx = 0 \quad \text{Mass conservation}
\]
Uniqueness is true only in $C([0, T]; L^1(\mathbb{R}^+, xdx)) \cap L^1(0, T; L^1(\mathbb{R}^+, x^{\gamma+1}dx))$. The fact that the solution is in $L^1(0, T; L^1(\mathbb{R}^+, xB(x)dx)$ is crucial.

For

$$B(x) = \frac{x}{2} (1 + x)^{-r}, \quad r \in (0, 1), \quad k_0 = 21_{[0,1]},$$

and for the initial condition $f_0 \in L^1(\mathbb{R}^+, xdx) \setminus L^1(\mathbb{R}^+, xB(x)dx)$ defined as

$$f_0(x) = \exp \left( - \int_0^x \frac{3 - ru(1 + u)^{-1}}{2\lambda(1 + u)^r + u} du \right),$$

there is a solution

$$f(t, x) = \exp(\lambda t)f_0(x)$$

belonging to $C([0, T); L^1(\mathbb{R}^+, (1 + x)dx))$ and for which the total mass of the system increases exponentially fast.
Large time behaviour of the fragmentation equation

**Theorem (Michel-Perthame-Escobedo-Mishler-Ricard – 2005)**

*Under reasonable technical assumptions, for large time, the profile \( f \) tends to distribute according to the self-similar profile \( g \). In other words, \( f(t, x) \) satisfies*

\[
f(t, x) \rightarrow t^{\frac{2}{\gamma}} g\left(x t^{\frac{1}{\gamma}}\right), \quad L^1(x \, dx)
\]

*where the self-similar profile \( g \) is the unique solution of*

\[
z \frac{dg}{dz}(z) + \left(2 + \alpha \gamma z^{\gamma}\right)g(z) = \alpha \gamma \int_{u=z}^{u=\infty} \frac{1}{u} k_0 \left(\frac{z}{u}\right) u^{\gamma} g(u) \, du, \quad \int_{0}^{\infty} zg(z)dz = \rho.
\]

- No admissible stationary state. Change of coordinates.
- In the new coordinates, the fragmentation equation rewrites as a transport/fragmentation equation.
- The moments of \( f \) for large \( t \) are then related to the moments of \( g \) via the following formula

\[
\int_{0}^{\infty} y^s g(y)dy \sim t^{(s-1)/\gamma} \int_{0}^{\infty} y^s f(t, y)dy, \quad s \in [0, \infty), \ t \ large.
\]
Two examples.

\[ k_0(x) = 2 \mathbb{1}_{[0,1]}(x) \]

Remark

*Mitosis is not included in the model. We exclude \( k_0(x) = 2 \delta_{x=1/2}(x) \).*
Inverse Problem (IP): Given a (noisy) measurement of $g(x)$ solution in $L^1((1 + x^\gamma + 1)dx)$ of the stationary equation, is it possible to estimate the functional parameters of the evolution equation, namely the triplet $(\alpha, \gamma, k_0) \in (0, +\infty) \times (0, +\infty) \times M^+(0, 1)$?
The fragmentation equation

Motivation : A new experiment

Uniqueness and for the inverse problem

Numerical simulations and application to real data
A new approach: To understand the fragmentation of Amyloid fibrils, MEASURE the time-evolution of the SIZE DISTRIBUTION

- **Proteins under study**: β2-Amyloid, Lysozyme, α-Synuclein, β−Lactoglobuline.

- **Step 1.** Prepare a pure sample of proteins.

- **Step 2.** Recreating the experimental conditions of mechanical fragmentation. The suspension is continuously agitated during two weeks by a magnetic agitator.

  From time to time, a small sample is taken out from the aliquot and saved.

- **Step 3.** A picture of each set taken out from the sample is done.

  Tool: Atomic Force Microscopy (AFM).

  Fibrils can be identified visually.
Step 4. **Determine** the localization and a **measure of the size of each fibril of the picture**.

Large fibrils are harder to detect than small fibrils:

- **a** Fibrils touching the border.
- **b** Fibrils which are crossing each other in a complicated way so that it is tricky to determine which is which.

⇝ It seems then that the number of large fibrils is underestimated.

Step 5. Estimate of much we lose and **modify the frequencies** by allocating a higher weight to the large sized fibrils. Very unusual technique since most experimentalists use directly the frequencies they measure as final data.
Application to real data

- We access **normalized cumulative distribution** function $h$ of the length distribution of each fibril sample.
Question: Determine $\gamma \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $k_0 \in \mathbb{R}^{N \times N}$.

- **Regularization of the data.** Approximate the data by polynomial functions.

- Make an **hypothesis on the fragmentation kernel** $k_0$: Parametrization.
  $\Rightarrow$ The problem becomes: Determine $\gamma, \alpha, k_1, k_2, k_3, k_4 \in \mathbb{R}^6$

- Solve the direct problem for the comprehensive set of admissible parameters $\gamma, \alpha, k_1, k_2, k_3, k_4 \in \mathbb{R}^6$.

- Determine which set of parameters gives rise to the dynamical behaviour observed experimentally. **Total linear least square analysis** (Minimization Problem).
1. The fragmentation equation

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4. Numerical simulations and application to real data
Tool: The Mellin transform

Definition

Let \( \mu \) be a measure over \( \mathbb{R} \). We denote by \( M[\mu] \) the Mellin transform of \( \mu \), defined by the integral

\[
M[\mu](s) = \int_{x=0}^{x=+\infty} x^{s-1} d\mu(x),
\]

for those values of \( s \) for which the integral exists.

Properties.

1. **Riemann-Lebesgue theorem.**
   If \( f \in L^1_{\text{loc}}(\mathbb{R}) \), then \( \lim_{v \to \pm \infty} F(u + iv) = 0. \)

2. If \( M[\mu](a) \) exists for some real number \( a,b \) then \( M[\mu] \) is **holomorphic** in the open bande \{ \( s \in \mathbb{C} | a < \text{Re} s < b \) \}.

3. The **inverse-Mellin** transform is defined by

\[
f(x) := \frac{1}{2i\pi} \int_{q^* - i\infty}^{q^* + i\infty} F(q^* + it)x^{-q^*} x^{-it} \, dt.
\]
Assumptions

- \( \exists \varepsilon > 0, \ 0 < \eta_1 < \eta_2 < 1 \) such that \( k_0(z) \geq \varepsilon, \ z \in [\eta_1, \eta_2] \),

- There exists \( \varepsilon > 0 \) such that \( k_0 \) is a bounded continuous function on \([1 - \varepsilon, 1] \) and on \([0, \varepsilon] \).

- There exists \( s_- < 2 \) such that \( K_0(s) \neq 1 \) for all \( s \in \mathbb{C} \) such that \( \Re(s) \in (s_-, 2) \).

- There exists \( h_0 \in L^1(0, 1) \setminus \{0\}, \ n \geq 0, \ a_j \in (0, 1) \) and \( C_j \geq 0 \), for \( j = 1 \ldots n \), such that
  \[
  k_0(x) = h_0(x) + \sum_{j=1}^{n} C_j \delta_{a_j}(x),
  \]
  where the \( a_j \) are such that there exists \( m_j \in \mathbb{Q}^n \) and \( \theta \) such that \( a_j = \theta^{m_j}, \ j = 1 \ldots n \).
The Mellin transform of $g$

**Proposition**

Suppose that a function $g$, satisfying

$$x^k g \in W^{1,1}(0, \infty) \quad \forall k \geq 0, \quad x^k g \in L^\infty(0, \infty), \quad \forall k \geq 1; \quad g \in W^{1,\infty}_{loc}(0, \infty)$$

is a solution of the stationary fragmentation equation in the sense of distributions on $(0, \infty)$, for some kernel $k_0 \in M^+(0, 1)$. Then, its Mellin transform:

$$G(s) = \int_0^\infty x^{s-1} g(x) dx$$

is well defined for all $s \in \mathbb{C}$ such that $\Re(s) \geq 1$, it is analytic in the domain

$$D_1 = \{ s \in \mathbb{C}; \Re(s) > 2 \}$$

and it satisfies:

$$(2 - s)G(s) = \alpha \gamma (K_0(s) - 1)G(s + \gamma) \quad \text{in} \quad D_1.$$
The stationary fragmentation equation in Mellin-coordinates: used to recover parameters

Define \( G(s) := M[g](s) \), \( K_0(s) := M[k_0](s) \) \((K_0(2) = 1)\).

Turn an integro-differential equation into a functional equation

\[
(2 - s)G(s) + \alpha \gamma G(s + \gamma) = \alpha \gamma K_0(s)G(s + \gamma), \quad s \in D_1. 
\]  

- Determine \( \gamma \).

- Determine \( \alpha \). Plug \( s = 2 \).

\[
\alpha = \frac{G(1)}{\gamma G(1 + \gamma)}. 
\]

- Determine \( K_0. \) \((\rightsquigarrow k_0)\)

\[
K_0(s) = 1 + \frac{G(s)(2 - s)}{\alpha \gamma G(s + \gamma)}, \quad s \in D_1. 
\]
Uniqueness of a triplet solution to the inverse problem

**Theorem (Doumic, Escobedo, T.)**

For all \( g \in L^1(\mathbb{R}^+) \) such that for all \( k \geq 0 \), \( \int_0^\infty y^k g(y)dy < \infty \), there exists at most one triplet \((\gamma, \alpha, k_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{M}(0,1)\) such that \( g \) is the solution of the stationary equation in the sense of distributions.

Proof of the uniqueness for \((\gamma, \alpha)\):

**Proof 1.** Relies on the estimates obtained by Balague, Canizo, Gabriel: There exists a constant \( C > 0 \) and a power \( p \geq 0 \) such that

\[
g(x) \sim_{x \to \infty} Cx^p e^{-\frac{\alpha}{\gamma} x^\gamma} \rightarrow \log \left( \frac{1}{g} \right) \sim_{x \to \infty} \frac{\alpha}{\gamma} x^\gamma,
\]

**Proof 2.** Relies on estimates we obtain for \( G(s) \) when \( \text{Re}(s) \) is large.

\[
G(s + \gamma) = \Phi(s) G(s), \quad \Phi(s) = \frac{2 - s}{\alpha \gamma (K_0(s) - 1)}.
\]

Both proofs rely strongly on computations on the moments of \( g \), and their behaviour when the power of the moment tends to infinity.
Uniqueness of $\gamma$: Proof based on the Mellin transform

Proposition (Necessary condition for $\gamma$)

Suppose that $g \in L^1_{loc}(\mathbb{R}^+)$ such that $\int_{\mathbb{R}^+} x^k g(x) dx < +\infty$, $k \geq 1$, and $g$ solves the stationary fragmentation equation for some parameters $\gamma > 0$, $\alpha > 0$ and some non negative measure $k_0$, compactly supported in $[0, 1]$. Let $G$ be the Mellin transform of $g$. Then, given any constant $R > 0$:

$$
\lim_{s \to \infty, s \in \mathbb{R}^+} \frac{s G(s)}{G(s + R)} = \begin{cases} 
0, & \forall R > \gamma \\
\alpha \gamma, & \text{if } R = \gamma \\
\infty, & \forall R \in (0, \gamma)
\end{cases}
$$

Idea of the proof: Equation satisfied by $G$ is close, for $s$ large, to equation

$$
AG(s + \gamma) = sG(s), \quad s \in \mathbb{R}, \quad G(2) = \rho.
$$

It has a unique analytical solution $\Gamma_{A, \gamma}(s) = K \left( \frac{\gamma}{A} \right)^{\frac{s}{\gamma}} \Gamma \left( \frac{s}{\gamma} \right)$,

$$
\Gamma_{A, \gamma}(s) \sim K \sqrt{\frac{2\pi \gamma}{s}} \left( \frac{\gamma}{A} \right)^{\frac{s}{\gamma}} \left( \frac{s}{\gamma} \right)^{\frac{s}{\gamma}} \left( 1 + O_{s \to \infty} \left( \frac{1}{s} \right) \right) = K \sqrt{\frac{2\pi \gamma}{s}} s^{-\frac{1}{2}} e^{\frac{s}{\gamma}} (\log s - 1 - \log A)
$$
Proof of the uniqueness for \( k_0 \)

Since \( G(s + \gamma) \) is strictly positive for \( s \in (-\gamma, +\infty) \) we deduce that for \( s \in [1, \infty) \) we can define \( K_0(s) \) by

\[
K_0(s) := 1 + \frac{G(s)(2 - s)}{\alpha \gamma G(s + \gamma)}, \quad 1 \leq s < +\infty.
\]

\[
K_0(s) := \int_0^1 x^{s-1} k_0(x) \, dx = \int_0^\infty e^{-sy} \mathcal{K}(y) \, dy = \hat{\mathcal{K}}(s)
\]

The Mellin transform \( K_0 \) can be interpreted as a Laplace transform \( \hat{\mathcal{K}} \) of the function \( \mathcal{K}(y) := k_0(e^{-y}) \) with \( y \in (0, +\infty) \), (if \( k_0 \) is diffuse) or of the pushforward \( T \# k_0 \) of \( k_0 \) by a function \( T \).

Using the uniqueness property of the Laplace transform for measures, we conclude that this defines uniquely \( \mathcal{K} \) and thus \( k_0 \).
Reconstruction of the fragmentation kernel

We have a constructive way to determine $\gamma$ and $\alpha$, but not $k_0$.

Idea : Define its Mellin transform using the formula

$$K_0^{\text{est}} := 1 + \frac{(2 - s)G(s)}{\alpha \gamma G(s + \gamma)}, \quad \Re(s) \geq 2$$  \hspace{1cm} (3)

To define $k_0$ from this formula, 2 points need to be solved:

- be allowed to divide by $G(s + \gamma)$, i.e. prove that $G$ does not cancel at least on a vertical strip of the complex plane,

- be allowed to define the inverse Mellin transform of $K_0^{\text{est}}$, and prove that this corresponds to the original $k_0$. 
Reconstruction of $k_0$

**Theorem**

Suppose that $g \in \bigcap_{k=1}^{\infty} L^1(x^k \, dx) \cap L_{loc}^1 (0, \infty)$ is the unique solution of the stationary equation for some given parameters $\alpha$ and $\gamma$ and where $k_0$ is a nonnegative measure, compactly supported in $[0, 1]$. Let $G(s)$ be the Mellin transform of the function $g$. Then, there exists $s_0 > 0$ such that

(i) $|G(s)| \neq 0$, $\forall s \in \mathbb{C}$; $\Re(s) \in [s_0, s_0 + \gamma]$,

(ii) $K_0(s) = 1 + \frac{(2 - s)G(s)}{\alpha \gamma G(s + \gamma)}$, for $\Re(s) = s_0$

(iii) $k_0(x) = \frac{1}{2i\pi} \int_{\Re(s)=s_0} x^{-s} \left(1 + \frac{(2 - s)G(s)}{\alpha \gamma G(s + \gamma)}\right) ds$. 

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Inverse Probleme
Direct problem. Goal: check that $G$ does not cancel.

Based on the Wiener-Hopf method, we can exhibit an explicit solution to the functional equation

\[
G(s + \gamma) = \Phi(s)G(s), \quad \Phi(s) = \frac{2 - s}{\alpha \gamma(K_0(s) - 1)}, \quad G(2) = \rho. \tag{4}
\]

Useful to

- check that the Mellin transform $G$ of the function $g$ never cancels,
- provide estimates on the inverse problem and derive a stability result.

Not useful to

- invert the problem, and to get the parameters from the solution.
Direct problem: Wiener Hopf Method 1/2

Solve \[ G(s + \gamma) = \Phi(s)G(s), \quad \Phi(s) = \frac{2 - s}{\alpha \gamma (K_0(s) - 1)}. \] \hspace{1cm} (5)

1. Transform the strip \( \{ z \in \mathbb{C} \mid s_0 < \text{Im}(z) < s_0 + \gamma \} \) into the exterior of \( \mathbb{R}^+ \)

We denote by \( F(\zeta) = G(s), \quad \tilde{\varphi}(\zeta) = \Phi(s), \quad \zeta \in \mathbb{C} \setminus \mathbb{R}^+ \),

The jump of the function \( F \) through the half line \( \mathbb{R}^+ \) satisfies

\[ F(x - i0) = \varphi(x)F(x + i0), \quad x \in \mathbb{R}^+, \]

where \( F(x + i0) := \lim_{\epsilon \to 0} F(xe^{i\epsilon}), \quad F(x - i0) := \lim_{\epsilon \to 0} F(xe^{i(2\pi - \epsilon)}), \)
2. **Change of variable** to obtain a "typical" Carleman equation (additive),

\[ P(x - i0) = \log(\varphi(x)) + P(x + i0), \quad x \in \mathbb{R}^+. \]

we look for a solution of the shape

\[ F(\zeta) = \exp(P(\zeta)), \quad \zeta \in \mathbb{C} \setminus \mathbb{R}^+. \]

3. **A candidate.** The function \( P \) is analytic in \( \mathbb{C} \setminus \mathbb{R}^+ \) and its jump through the half-line \( \mathbb{R}^+ \) is determined by the Carleman equation. A candidate for the function \( P \) is the function

\[ P(\zeta) = -\frac{1}{2i\pi} \int_0^{+\infty} \log(\varphi(w)) \left\{ \frac{1}{w - \zeta} - \frac{1}{w + 2} \right\} dw, \quad \zeta \in \mathbb{C} \setminus \mathbb{R}^+. \]

4. **Back to \( F \) and \( G \)**

\[ \overline{F}(\zeta) = \exp \left( -\frac{1}{2i\pi} \int_0^{+\infty} \log(\varphi(w)) \left\{ \frac{1}{w - \zeta} - \frac{1}{w + 2} \right\} dw \right) \rightarrow \overline{G}. \]
5. **Check** that $\overline{G} \equiv G$.

Both functions $\overline{G}$ and $G$ = the Mellin transform of $g$ are analytic and satisfy the equation (5), but nothing guarantees that $G = \overline{G}$.

Indeed, for $\Phi(s) = s$ and $\gamma = 1$, the functions

$$s \mapsto \frac{1}{2} \Gamma(s) \quad \text{and} \quad s \mapsto \frac{1}{2} \Gamma(s) \left[1 + \sin(2\pi s)\right]$$

are two distinct solutions.

**Theorem**

*Let $g$ be the solution to the stationary equation, and $\overline{G}$ the function defined in (6). Then*

$$g(x) = \frac{1}{2i\pi} \int_{\Re(s)=u} \overline{G}(s)x^{-s} \, ds, \quad \forall u > s_0. \quad (7)$$
The fragmentation equation

Motivation: A new experiment

Uniqueness and for the inverse problem

Numerical simulations and application to real data
Determination of $\gamma$

Recall the Mellin transform of $f$

$$M[f(t, .)](s + 1) = \int_0^\infty x^s f(t, x)\,dx.$$  

Assuming that the distribution $f$ has reached equilibrium for a given time $t$,  

$$M[f(t, .)](s + 1) = \int_0^{\infty} x^s t^{2/\gamma} g(t^{1/\gamma} x)\,dx = t^{(1-s)/\gamma} \int_0^\infty y^s g(y)\,dy, \quad s \in \mathbb{R}.$$  

Then, there exists a constant $C_s$ such that

$$\log (M[f(t, .)](s + 1)) = \frac{1-s}{\gamma} \log(t) + \log(C_s).$$

Then, we expect that for any given $s \in [0, +\infty]$, and for $t$ large enough,

the function $l_s : \log(t) \mapsto \frac{\log (M[f(t, .)](s + 1))}{1 - s}$ has a slope of $1/\gamma$.
Plots of the function $I_s$ at different times and for $s \in [1, 6.2]$. 

Simulated data. Determination of $\gamma$
Simulated data. Determination of $\gamma$

$\gamma = 1$

$\gamma = 2$
Simulated data. Determination of $\alpha$

The value for $\alpha$ is directly obtained as a function of $\gamma$ and $G$ (which depends on $\gamma$) by using $K_0(1) = 2$

$$\alpha_{est} = \frac{G(1)}{\gamma_{est} G(1 + \gamma_{est})}.$$

Uniform kernel

Gaussian kernel
Simulated data. Determination of $k_0$

$$K_{0}^{\text{est}}(s) = 1 + \frac{(2 - s)G(s)}{\alpha_{\text{est}} \gamma_{\text{est}} G(s + \gamma_{\text{est}})}, \quad \text{Re}(s) > 2,$$

$$k_{0}^{\text{est}}(x) = \frac{1}{2\pi} \int_{u-i\infty}^{u+i\infty} K_{0}^{\text{est}}(s)x^{-s} \, ds.$$
Simulated data. Determination of $k_0$

Complicated to estimate $k_0$ using the Mellin inverse formula.

**Questions of interest** Does $k_0$ charge the boundaries, or the center of the fibrils? If the center is more likely to break apart, what is the spread of the gaussian?

**What we do** Assume $k_0^{a,b,\eta}(x) = a1_{[\eta,1-\eta]} + b1_{([0,\eta] \cup [1-\eta,\eta])}$, and find $a$, $b$ and $\eta$ such that $\|K_{est}^0 - K_0^{a,b,\eta}\|_{L^2(1+i[-1,1])}$ is minimal.
Real data: Determination of $\gamma$

Plots of the function $I_s$ at different times and for $s \in [1, 6.2]$.

Data are noisy: Should we take into account all the data points to predicts $\gamma$?
Real data: Determination of $\gamma$

$\beta$-lactamase

$\alpha$-synuclein
Real data: Determination of $\alpha$
Test case: $\beta$2-amyloid

Comparison between the value we predict for $B(x)$ (left) and what the biologists found (right).
Conclusion and Perspectives

- Well posedness of the inverse problem
- Determination of a constructive and non-parametric method to recover the parameters
- Test of our method on simulated data
- Need to determine a protocol to deal with real data.

Thank you!