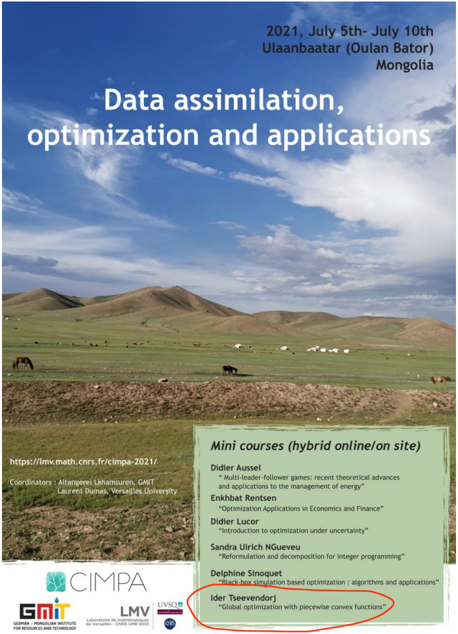
2021, July 5th- July 10th
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Data assimilation,
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"Global optimization with piecewise convex functions"
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Global optimization with piecewise convex functions Lecture 1

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Outline

- A function $f(\cdot)$ is called convex if the line segment between any two points on the graph of the function lies above the graph between the two points i.e.

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all α such that $0 \leq \alpha \leq 1$ and for all $x, y \in \mathbb{R}^n$.

- A function $f(\cdot)$ is called concave if $-f(\cdot)$ is convex.

Convex sets

- A subset D of real vector space is called convex if D contains every line segment whose endpoints belong to D, i.e.

$$\alpha x + (1 - \alpha) y \in D \text{ for all } x, y \in D.$$

Optimization problem

An optimization problem is the problem of finding the best solution from all feasible solutions.

Claim (Special form)

Any optimization problem can be written in the following form :

$$\left\{\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in S, \\ & g(x) \leq 0, \end{array}\right. \tag{P}$$

where $x = (x_1, ..., x_n)^\top$, S is a nonempty simple set in $\mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuous functions.

Optimization problem

Standard form of optimization

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & h_i(x) = 0, j = 1, ..., r \\ & g_j(x) \leq 0, i = r+1, ..., m \end{cases}$$

$$S = \mathbb{R}^n; g(x) = max\{h_1(x), ..., h_r(x), -h_1(x), ..., -h_r(x), g_{r+1}, ..., g_m(x)\}$$

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & h_i(x) = 0, j = 1, ..., r \\ & g_j(x) \leq 0, i = r+1, ..., m \end{cases} \Leftrightarrow \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in S, \\ & g(x) \leq 0 \end{cases}$$

Exercice 1. (linear programming)

Write the linear programming problem

$$\begin{cases} \text{maximize} & \langle c, x \rangle \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{cases}$$

in the special form of (P).
$f(\cdot) = ?, S = ?, g(\cdot) = ?$

Optimization problem

D is a domain (all feasible solutions)

$$D = \{x \in \mathbb{R}^n \mid x \in S, g(x) \leq 0\}$$

$f(\cdot)$ is the objective function

$$\left\{ \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in S, \\ & g(x) \leq 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in D. \end{array} \right. \qquad (P)$$

Optimal solutions

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in D. \end{cases} \qquad (P)$$

Local solution

A local solution y of (P) is defined as an element of D for which there exists some $\delta > 0$ such that for all $x \in D$ where $\|x - y\| \leq \delta$ the expression

$$f(y) \leq f(x)$$

holds.

Global solution

An element z of D is called the global solution of (P) if

$$f(z) \leq f(x) \text{ for all } x \in D.$$

Convex optimization

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in S, \\ & g(x) \leq 0 \end{cases} \qquad (P)$$

Definition

The problem (P) is called **convex** iff the following four conditions hold :

1. minimization problem,
2. f is convex function,
3. S is convex set,
4. g is convex function.

Convex optimization

Optimization problem

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in S, \\ & g(x) \leq 0 \end{cases} \qquad (P)$$

Properties

- S, g are convex $\Longrightarrow$
  Domain $D = \{x \in \mathbb{R}^n \mid x \in S, g(x) \leq 0\}$ is convex.
- If the problem (P) is convex then any local solution is the global solution.

Optimization problem

Exercice 2.

Show that the linear programming problem

$$\begin{cases} \text{maximize} & \langle c, x \rangle \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{cases}$$

is convex after the definition of the convex optimization by the special form of (P).

Nonconvex optimization

Optimization problem

$$\left\{ \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in S, \\ & g(x) \leq 0 \end{array} \right. \qquad (P)$$

Definition

If at least one of the four conditions is violated

1. minimization problem,
2. f is convex function,
3. S is convex set,
4. g is convex function.

then the optimization problem (P) can be **Nonconvex**.

Classification of Nonconvex problems

Definition

If at least one of the four conditions is violated

1. minimization problem,
2. f is convex function,
3. S is convex set,
4. g is convex function.

then the optimization problem (P) can be Nonconvex.

Convex maximization

Condition 1) is violated $\Longrightarrow$

$$\begin{cases} \text{maximize} & f(x) \\ \text{subject to} & x \in D. \end{cases} \qquad (CM)$$

Classification of Nonconvex problems

Definition

If at least one of the four conditions is violated

1. minimization problem,
2. f is convex function,
3. S is convex set,
4. g is convex function.

then the optimization problem (P) can be Nonconvex.

Reverse Convex Minimization

Condition 4) is violated

$g(x) \leq 0$ with $g(x)$ concave $\Leftrightarrow g_1(x) \geq 0$ with $g_1(x) = -g(x)$ convex

$$\Longrightarrow \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in S \\ & g_1(x) \geq 0 \end{cases} \qquad (RCP)$$

Classification of Nonconvex problems

Definition

If at least one of the four conditions is violated

1. minimization problem,
2. f is convex function,
3. S is convex set,
4. g is convex function.

then the optimization problem (P) can be Nonconvex.

DC optimization

Condition 2) is violated
$f(x) = f_1(x) - f_2(x)$ difference of two convex functions $\Longrightarrow$

$$\begin{cases} \text{maximize} & f_1(x) - f_2(x) \\ \text{subject to} & x \in D \end{cases} \qquad (DC)$$

Global Optimization

Definition

If at least one of the four conditions is violated

1. minimization problem,
2. f is convex function,
3. S is convex set,
4. g is convex function.

then the optimization problem (P) can be nonconvex.

Global optimization

Condition 2) is violated

$f(x)$ any continious nonconvex function $\Longrightarrow$ Global Optimization

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in D \end{cases} \quad \text{or} \quad \begin{cases} \text{maximize} & f(x) \\ \text{subject to} & x \in D \end{cases} \qquad (GO)$$

Why Global Optimization ?

- Nonconvex optimization problems can have a large number of local minima which makes the problem of finding the global solution difficult.
- Finding the global minimum of a function is far more difficult: analytical methods are frequently not applicable, and the use of numerical solution strategies often leads to very hard challenges.

Piecewise convex function

Definition

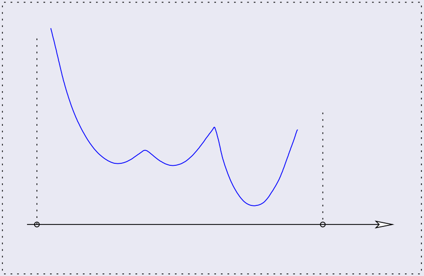
A function $F : \mathbb{R}^n \to \mathbb{R}$ is called a piecewise convex function if it can be decomposed into :

$$F(x) = min\{f_1(x), f_2(x), ..., f_m(x)\},$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$ are convex functions for all $j \in M = \{1, 2..., m\}$.

Piecewise convex function

Convex functions | **Piecewise convex function**



Space of Piecewise Convex Functions

Remark

Notice first that real valued convex and concave functions are particular cases of piecewise convex functions,
since a piecewise convex function is

- convex when there is only one convex piece $m = 1$;
- concave when all functions f_j are affine.

Space of Piecewise Convex Functions

Proposition 1 (Continuity of Piecewise Convex Function)

A function

$$F(x) = min\{f_j(x) \mid j \in M\}$$

is continuous, if

- each f_j is continuous and
- the index set M is finite

Operations on Piecewise Convex Functions

Proposition 2

Let $G(\cdot), H(\cdot)$ be two piecewise convex functions, then

- $G + \alpha$ for $\alpha \in \mathbb{R}$,
- $G + H$,
- λG for $\lambda > 0$,
- $G^+ = max\{0, G\}$,
- $G^- = min\{0, G\}$,
- $max\{G, H\}$,
- $min\{G, H\}$,

are also piecewise convex.

Operations on Piecewise Convex Functions

Let two functions' representations be

$$G(x) = min\{g_i(x) \mid i = 1, ..., n\}, H(x) = min\{h_j(x) \mid j = 1, ..., m\}.$$

Proof of Proposition 2

- $G(x) + \alpha = F(x) = min\{f_k(x) \mid k = 1, ..., n\}$ with $f_k(x) = g_k(x) + \alpha$;
- **addition** $(G + H)(x) = F(x) = min\{f_k(x) \mid k = 1, ..., n \times m\}$ with $f_k(x) = f_{ij}(x) = g_i(x) + h_j(x)$;
- **positive scalar multiplication** $\lambda G(x) = F(x) = min\{f_k(x) \mid k = 1, .., n\}$, with $f_k(x) = \lambda g_k(x)$;
- **operation "min"** $min\{G(x), H(x)\} = F(x) = min\{f_k(x) \mid k = 1, .., n + m\}$ with

$$f_k(x) = \begin{cases} g_k(x) & k = 1, .., n \\ h_{k-n}(x) & k = n + 1, ..., n + m \end{cases}$$

In all cases, functions $f_k(\cdot)$ are convex, therefore $F(x)$ is piecewise convex.

Exercice 3

Prove that $G^{+}(x), G^{-}(x)$ are piecewise convex.

Exercice 4 (Homework)

Prove that $max\{G(x), H(x)\}$ is a piecewise convex and give its rule of calcul.

Piecewise affine approximation

Let us consider a concave function $\phi(\cdot)$.

Affine majorant

For the concave function $\phi(x)$ at any y there is an affine majorant $\ell_y(x)$ such that

$$\begin{cases} \phi(x) & \leq \ell_y(x) \text{ for all } x \in \mathbb{R}^n \\ \phi(y) & = \ell_y(y) \end{cases}$$

Example of affine majorant

If $\phi(x)$ is differentiable then due to the following inequality $\phi(x) \leq \phi(y) + \langle \nabla(y), x - y \rangle$ for all x, y we have

$$\ell_y(x) = \phi(y) + \langle \nabla\phi(y), x - y \rangle$$

Piecewise affine approximation

Affine majorants

A set of points $y^1, \dots y^N$ gives us affine majorants $\ell_1(x), \dots, \ell_N(x)$.

Piecewise affine majorant

Any concave function $\phi(\cdot)$ can be approximated by a piecewise affine function $L(\cdot)$:

$$\phi(x) \approx L(x) = min\{\ell_j(x) \mid j = 1, ..., N\}$$

such that

$$\begin{cases} \phi(x) & \leq L(x) \text{ for all } x \in \mathbb{R}^n \\ \phi(y^j) & = L(y^j) \text{ for all } j = 1, \dots N \end{cases}$$

Piecewise affine approximation

$$\phi(x) \approx L(x) = min\{\ell_j(x) \mid j = 1, ..., N\}$$

Powerful tool in convex optimization

$$\begin{cases} \text{maximize} & \phi(x) \\ \text{subject to} & x \in D \end{cases} \qquad \begin{cases} \text{minimize} & -\phi(x) \\ \text{subject to} & x \in D \end{cases}$$

where $\phi(\cdot)$ is concave, D is convex.

From Affine Majorant to Convex Majorant

Remark

Unfortunately, piecewise affine approximation does not work for nonconvex functions, so our aim is to extend affine approximation to convex approximation to deal with nonconvex cases.

Convex Majorant

Let us consider a nonconvex function $\psi(\cdot)$.

Definition

A convex function $f_y(x)$ is a convex majorant to a nonconvex function $\psi(x)$ at y iff

$$\begin{cases} \psi(x) & \leq f_y(x) \text{ for all } x \in \mathbb{R}^n \\ \psi(y) & = f_y(y) \end{cases}$$

Classes of functions that have convex majorant

Class of DC functions

For a function decomposed as a difference of convex functions $\psi(x) = f_1(x) - f_2(x)$, its convex majorant is

$$f_y(x) = f_1(x) - (f_2(y) + \langle \nabla f_2(y), x - y \rangle);$$

Class of Lipschitz functions

Lipschitz functions with constant L:

$$| \psi(x) - \psi(y) | \leq L \, \|x - y\| \;\text{ for all } x, y \in \mathbb{R}^n$$

possess a convex majorant also.

Classes of functions that have convex majorant

Exercice 5.

Find a convex majorant for a Lipschitz function.

Global Optimization Problem

Nonconvex optimization problem

Let us consider the following nonconvex problem :

$$\begin{cases} \text{maximize} & \psi(x) \\ \text{subject to} & x \in D \end{cases} \qquad (GO)$$

where $D \subset \mathbb{R}^n$ is convex, $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous and nonconvex function.

Assumption 1

Let us assume that at any $y \in D$ there exists a convex majorant $f_y(\cdot)$ to $\psi(\cdot)$.

Global Optimization Problem

Initialization

Suppose that $y^0 \in D$ be a starting point.
Denote by $f_k(x)$ a convex majorant $f_{y_k}(x)$ to $\psi(x)$ for $k = 0, 1, ...$

Method

`Approximation(` ψ, D, y^0 `)`

1. `Set` $F_0(x) = f_0(x)$ `and` $k = 1$ `;`
2. `Define` y^k `as a solution to`

$$\begin{cases} \texttt{maximize} & F_{k-1}(x) \\ \texttt{subject to} & x \in D \end{cases} \qquad (PCMP)$$

3. `Set` $f_k(x) = f_{y_k}(x)$ `and`

$$F_k(x) = min\{F_{k-1}(x), f_k(x)\};$$

4. `if` $(F_{k-1}(y^k) - F_k(y^k)) \leq \varepsilon$ `then STOP else` $k = k + 1$`; Goto 2`

Assumption 2 (Piecewise Convex Maximization Problem)

Let us assume that the piecewise convex maximization problem ($PCMP$) can be solved for each iteration $k = 1, 2, ...$

Proposition

Let D be a compact set, $\psi(\cdot)$ be a continuous function on D and *Assumption 1, Assumption 2* hold.

Then for a sequence $\{y^k\}$ generated by the above method

$$i) \qquad \lim_{k \to +\infty} \psi(y^k) = \lim_{k \to +\infty} F_{k-1}(y^k) = \zeta^*$$

with an estimation $0 \leq \zeta^* - \psi(y^k) = F_{k-1}(y^k) - \psi(y^k)$;

$$ii) \qquad \lim_{k \to +\infty} d(y^k, Z^*) = 0.$$

where ζ^* and Z^* stand for the global maximal value and the set of global optimal solutions of ($PCMP$).

Sketch of the proof

By definition, we have $F_0(x) \geq \psi(x)$ for all $x \in D$;
thus $F_0(y^1) \geq \psi(y^1)$ and

$$max\{F_0(x) \mid x \in D\} \geq \zeta^* = max\{\psi(x) \mid x \in D\}.$$

By Assumption 1. there exists $f_1(\cdot)$, such that

$$\begin{cases} \psi(x) & \leq f_1(x) \text{ for all } x \in D \\ \psi(y^1) & = f_1(y^1) \end{cases}$$

Now, for any $x \in D$

$$\begin{cases} f_1(x) & \geq \psi(x) \\ F_0(x) & \geq \psi(x) \end{cases} \Longrightarrow F_1(x) = min\{f_1(x), F_0(x)\} \geq \psi(x).$$

Global Optimization Problem

Sketch of the proof

By definition, we get $F_0(x) \geq F_1(x) \geq \ldots F_k(x) \geq \psi(x)$ for all x. Thus,

$$F_{k-1}(y^k) \geq F_k(y^k) \geq \zeta^* \geq \psi(y^k) \Longrightarrow 0 \leq \zeta^* - \psi(y^k) = F_{k-1}(y^k) - \psi(y^k)$$

Numerical sequence $F_{k-1}(y^k)$ is decreasing and bounded below, that proves the existence of a limit :

$$\lim_{k \to +\infty} F_{k-1}(y^k).$$

The compactness of D provides a subsequence $y^{k_s} \to z$.
For all $l = 1, \ldots k-1$, $F_k(y^l) = \psi(y^l)$ and for all $x \in D$, $F_k(x) \geq \psi(x)$.

$$0 \leq \lim_{s \to +\infty} (\zeta^* - \psi(y^{k_s})) \leq \lim_{s \to +\infty} (F_{k_s}(y^{k_s}) - \psi(y^{k_s})) = 0 \Rightarrow \zeta^* = \psi(z).$$