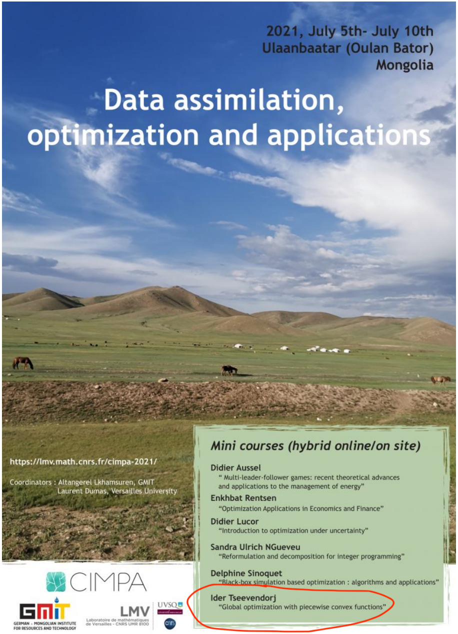
2021, July 5th- July 10th
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Data assimilation,
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https://lmv.math.cnrs.fr/cimpa-2021/
Coordinators : Altangerel Lkhamsuren, GMIT
Laurent Dumas, Versailles University
Mini courses (hybrid online/on site)
Didier Aussel
" Multi-leader-follower games: recent theoretical advances and applications to the management of energy"
Enkhbat Rentsen
"Optimization Applications in Economics and Finance"
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"Reformulation and decomposition for integer programming"
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Global optimization with piecewise convex functions Lecture 2

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July 7, 2021

Outline

EXERCICES from Lecture 1

Operations on Piecewise Convex Functions

$$G(x) = min\{g_i(x) \mid i = 1, ..., n\},$$

Exercice 3

Prove that $G^+(x)$, $G^-(x)$ are piecewise convex.

Proofs

- $G^+(x) = ?$

$$G^+(x) = \max\{0, G(x)\} = \max\{g_1^+(x), g_2^+(x), ..., g_n^+(x)\},$$

where $g_i^+(x)$ are convex.

- $G^-(x) = ?$

$$G^-(x) = \min\{0, G(x)\} = \min\{0, g_1(x), ..., g_n(x)\}.$$

$$G(x) = min\{g_i(x) \mid i = 1, ..., n\}, H(x) = min\{h_j(x) \mid j = 1, ..., m\}.$$

Exercice 4 (Homework)

Prove that $max\{G(x), H(x)\}$ is a piecewise convex and give its rule of calcul.

Rule

$$
\begin{aligned}
max\{G(x), H(x)\} = & \\
max & \quad \{min\{g_i(x) \mid i = 1, ..., n\}, min\{h_j(x) \mid j = 1, ..., m\}\} \\
min & \quad \{max\{g_i(x), h_j(x)\} \mid i = 1, ..., n, j = 1, ..., m\}
\end{aligned}
$$

$n \times m$

$n \times m$

$max\{g_i(x), h_j(x)\}$ are convex.

LECTURE 2

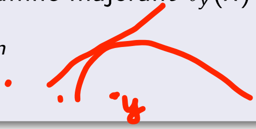
Piecewise affine approximation

Let us consider a concave function $\phi(\cdot)$.

Affine majorant

For the concave function $\phi(x)$ at any y there is an affine majorant $\ell_y(x)$ such that

$$\begin{cases} \phi(x) & \leq \ell_y(x) \text{ for all } x \in \mathbb{R}^n \\ \phi(y) & = \ell_y(y) \end{cases}$$

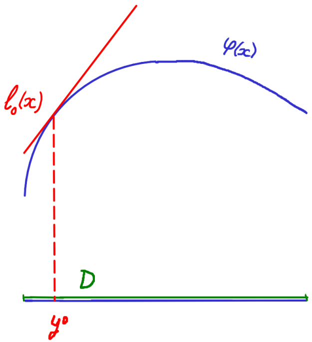


Example of affine majorant

If $\phi(x)$ is differentiable then due to the following inequality $\phi(x) \leq \phi(y) + \langle \nabla(y), x - y \rangle$ for all x, y we have

$$\ell_y(x) = \phi(y) + \langle \nabla \phi(y), x - y \rangle$$

Affine majorant



$$\begin{cases} \varphi(x) \le \ell_0(x) & \forall x \\ \varphi(y^0) = \ell_0(y^0) \end{cases}$$

ℓ_0 - affine majorant

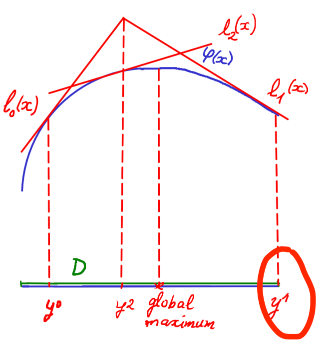
Piecewise affine approximation

Piecewise affine approximation in convex optimization

$$\begin{cases} \text{maximize} & \phi(x) \\ \text{subject to} & x \in D \end{cases}$$

where $\phi(\cdot)$ is concave, D is convex.

Solving convex optimization by Piecewise affine approximation



$y^0 \to \ell_0(x)$

$y^1 \quad \begin{cases} \max\limits_{x \in D} \ell_0(x) \end{cases} \to \ell_1(x)$

$y^2 \quad \begin{cases} \max\limits_{x \in D} \min\{\ell_0(x), \ell_1(x)\} \end{cases} \to \ell_2(x)$

$\dots$

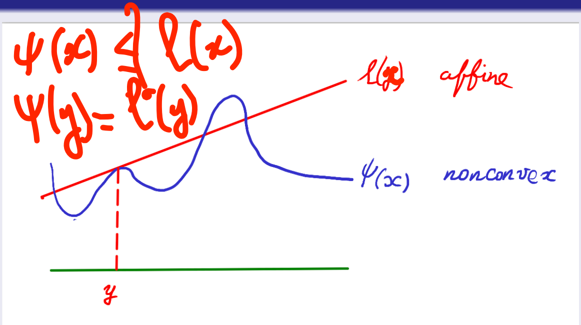
$y^k \quad \begin{cases} \max\limits_{x \in D} \min\{\ell_i(x) / i = 0, \dots k-1\} \end{cases}$

$\{y^k\} \to$ global maximum of $\begin{cases} \max\limits_{x \in D} \varphi(x) \end{cases}$

From Affine Majorant to Convex Majorant

Remark

Unfortunately, piecewise affine approximation does not work for nonconvex functions, so our aim is to extend affine approximation to convex approximation to deal with nonconvex cases.



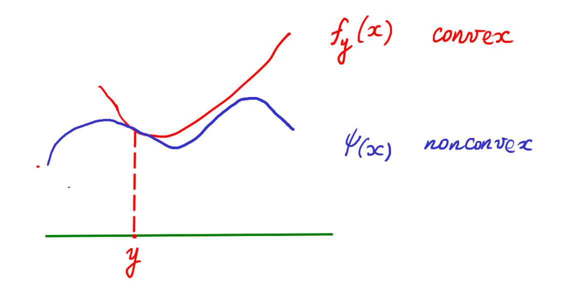
Convex Majorant

Let us consider a nonconvex function $\psi(\cdot)$.

Definition

A convex function $f_y(x)$ is a convex majorant to a nonconvex function $\psi(x)$ at y iff

$$\begin{cases} \psi(x) & \leq f_y(x) \text{ for all } x \in \mathbb{R}^n \\ \psi(y) & = f_y(y) \end{cases}$$



Classes of functions that have convex majorant

Class of DC functions

For a function decomposed as a difference of convex functions $\psi(x) = f_1(x) - f_2(x)$, its convex majorant is

$$f_y(x) = f_1(x) - (f_2(y) + \langle \nabla f_2(y), x - y \rangle);$$

convex

Class of Lipschitz functions

Lipschitz functions with constant L:

$$|\ \psi(x) - \psi(y)\ | \leq L\ \|x - y\| \ \text{ for all } x, y \in \mathbb{R}^n$$

possess a convex majorant also.

Exercice 5.

Find a convex majorant for a Lipschitz function.

Global Optimization Problem

Nonconvex optimization problem

Let us consider the following nonconvex problem :

$$\left\{ \begin{array}{ll} \text{maximize} & \psi(x) \\ \text{subject to} & x \in D \end{array} \right. \qquad (GO)$$

where $D \subset \mathbb{R}^n$ is convex, $\psi : \mathbb{R}^n \to \mathbb{R}$ is a continuous and nonconvex function.

Assumption 1

Let us assume that at any $y \in D$ there exists a convex majorant $f_y(\cdot)$ to $\psi(\cdot)$.

Global Optimization Problem

Initialization

Suppose that $y^0 \in D$ be a starting point.
Denote by $f_k(x)$ a convex majorant $f_{y_k}(x)$ to $\psi(x)$ for $k = 0, 1, ...$

Method

`Approximation(` ψ, D, y^0 `)`

1. `Set` $F_0(x) = f_0(x)$ `and` $k = 1$ `;`
2. `Define` y^k `as a solution to`

$$\begin{cases} \texttt{maximize} & F_{k-1}(x) \\ \texttt{subject to} & x \in D \end{cases} \qquad (PCMP)$$

3. `Set` $f_k(x) = f_{y_k}(x)$ `and`

$$F_k(x) = min\{F_{k-1}(x), f_k(x)\};$$

4. `if` $(F_{k-1}(y^k) - F_k(y^k)) \leq \varepsilon$ `then STOP else` $k = k+1$`; Goto 2`

Global Optimization Problem

Assumption 2 (Piecewise Convex Maximization Problem)

Let us assume that the piecewise convex maximization problem $(PCMP)$ can be solved for each iteration $k = 1, 2, \ldots$

Proposition

Let D be a compact set, $\psi(\cdot)$ be a continuous function on D and *Assumption 1, Assumption 2* hold.
Then for a sequence $\{y^k\}$ generated by the above method

$$i) \qquad \lim_{k\to+\infty} \psi(y^k) = \lim_{k\to+\infty} F_{k-1}(y^k) = \zeta^*$$

with an estimation $0 \leq \zeta^* - \psi(y^k) = F_{k-1}(y^k) - \psi(y^k)$;

$$ii) \qquad \lim_{k\to+\infty} d(y^k, Z^*) = 0.$$

where ζ^* and Z^* stand for the global maximal value and the set of global optimal solutions of $(PCMP)$.

Global Optimization Problem

Sketch of the proof

By definition, we have $F_0(x) \geq \psi(x)$ for all $x \in D$;
thus $F_0(y^1) \geq \psi(y^1)$ and

$$max\{F_0(x) \mid x \in D\} \geq \zeta^* = max\{\psi(x) \mid x \in D\}.$$

By Assumption 1. there exists $f_1(\cdot)$, such that

$$\begin{cases} \psi(x) & \leq f_1(x) \text{ for all } x \in D \\ \psi(y^1) & = f_1(y^1) \end{cases}$$

Now, for any $x \in D$

$$\begin{cases} f_1(x) & \geq \psi(x) \\ F_0(x) & \geq \psi(x) \end{cases} \Longrightarrow F_1(x) = min\{f_1(x), F_0(x)\} \geq \psi(x).$$

Global Optimization Problem

Sketch of the proof

By definition, we get $F_0(x) \geq F_1(x) \geq \dots F_k(x) \geq \psi(x)$ for all x. Thus,

$$F_{k-1}(y^k) \geq F_k(y^k) \geq \zeta^* \geq \psi(y^k) \Longrightarrow \boxed{0 \leq \zeta^* - \psi(y^k) = F_{k-1}(y^k) - \psi(y^k)}$$

Numerical sequence $F_{k-1}(y^k)$ is decreasing and bounded below, that proves the existence of a limit :

$$\lim_{k \to +\infty} F_{k-1}(y^k).$$

The compactness of D provides a subsequence $y^{k_s} \to z$. $\in D$
For all $l = 1, \dots k-1$, $F_k(y^l) = \psi(y^l)$ and for all $x \in D$, $F_k(x) \geq \psi(x)$.

$$0 \leq \lim_{s \to +\infty} (\zeta^* - \psi(y^{k_s})) \leq \lim_{s \to +\infty} (F_{k_s}(y^{k_s}) - \psi(y^{k_s})) = 0 \Rightarrow \zeta^* = \psi(z).$$

Piecewise convex maximization :

We are given a convex compact D and we consider the following problem:

$$\left\{\begin{array}{ll} \text{maximize} & F(x) \\ \text{subject to} & x \in D \end{array}\right. \qquad (PCMP)$$

This problem is called piecewise convex maximization (PCMP) when its objective function $F(\cdot)$ is piecewise convex.

The purpose of this part

is to establish necessary and sufficient optimality condition for the piecewise convex maximization problem $(PCMP)$.

Convex Maximization :

First, we consider the well known convex maximization problem :

$$\begin{cases} \text{maximize} & f(x) \\ \text{subject to} & x \in D \end{cases} \qquad (CM)$$

where D is convex compact and $f(\cdot)$ is a convex function.

Remark

It is clear that (CM) is a particular case of $(PCMP)$ when $m = 1$ (there is only one piece).

Lebesgue's set of a function $f(\cdot)$ at level α

$$\mathcal{L}_f(\alpha) = \{x \mid f(x) \leq \alpha\}.$$

Given a point $z \in D$ and a continuous function $f(\cdot)$ for maximizing.

Observation

Lebesgue's set $\mathcal{L}_f(f(z))$ of a function $f(\cdot)$ at level $f(z)$ contains all points no better than z.

$$\mathcal{L}_f(f(z)) = \{x \mid f(x) \leq f(z)\}$$

Exercice 6

Show that Lebesgue's set of a convex function $f(\cdot)$

$$\mathcal{L}_f(\alpha) = \{x \mid f(x) \leq \alpha\}$$

is convex at any level $\alpha \in \mathbb{R}$.

Global Optimality Conditions

Clearly, the point z is a global maximum of (CM) iff all points of the domain D are simply no better than z.

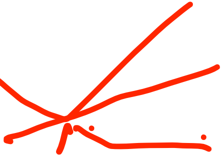
It means that $z \in D$ solves (CM) globally if and only if

$$\boxed{D \subset \mathcal{L}_f(f(z)).}$$

One should check inclusion of two convex sets !

Subdifferential, Normal cone

Let $f(\cdot)$ be a convex function.



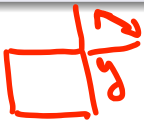
Definition (Subdifferential)

$$\partial f(y) = \{y^* \mid f(x) - f(y) \geq \langle y^*, x - y \rangle \text{ for all } x$$

Let D be a convex set.

Definition (Normal cone)

$$N(D, y) = \{y^* \mid \langle y^*, x - y \rangle \leq 0 \text{ for all } x \in D.$$



Theorem [Journal of the Mongolian Math. Society, 1998]

Assume that there exists a $v \in \mathbb{R}^n$ such that $f(v) < f(z)$ for a feasible point z.
Then a necessary and a sufficient condition for $z \in D$ to be a global maximum for (CM) is:

$$\begin{array}{c} \partial f(y) \bigcap N(D, y) \neq \{0\} \\ \text{for all } y \text{ such that } f(y) = f(z). \end{array} \tag{gNS}$$

Proof (necessary)

Let $z \in D$ solve (CM) globally, in other words

$$f(z) \geq f(x) \text{ for all } x \in D.$$

Then for all y such that $f(y) = f(z)$ the following chain

$$0 \geq f(x) - f(z) = f(x) - f(y) \geq \langle y^*, x - y \rangle$$

hold for all $y^* \in \partial f(y)$, $y^* \neq 0$ and for all $x \in D$.
The chain implies

$$y^* \in \partial f(y) \cap N(D, y).$$

Proof (sufficient)

Suppose that the condition (gNS) holds, but z is not the global maximum of (CM). Thus, there is a point $u \in D$ such that $f(u) > f(z)$.
We consider a segment between two points u, v,

$$y(\alpha) = \alpha v + (1-\alpha)u, \qquad 0 \le \alpha \le 1$$

where v is the point assumed to exist such that $f(z) > f(v)$.
$f(u) > f(z) > f(v)$ implies that there is $\alpha_0 \in]0,1[$ such that $f(y(\alpha_0)) = f(z)$.

$$\langle y_0^*, u - y(\alpha_0)\rangle = \left\langle y_0^*, \frac{y(\alpha_0)-\alpha_0 v}{1-\alpha_0} - y(\alpha_0)\right\rangle$$

$$\ge \frac{\alpha_0}{1-\alpha_0}(f(v) - f(y(\alpha_0))) > 0$$

proves that $y_0^* \notin N(D, y(\alpha_0))$ for all $y_0^* \in \partial f(y(\alpha_0)) \setminus \{0\}$.

Exercice 7

We consider (CM) in $\mathbb{R}^2$ with

$$f(x_1, x_2) = max\{2x_1 + 3x_2, 3x_1 - x_2, -2x_1 + x_2, -2x_1 - 6x_2\}$$

$$D = \{x \in \mathbb{R}^2 \mid -3 \leq x_i \leq 3, i = 1, 2\}$$

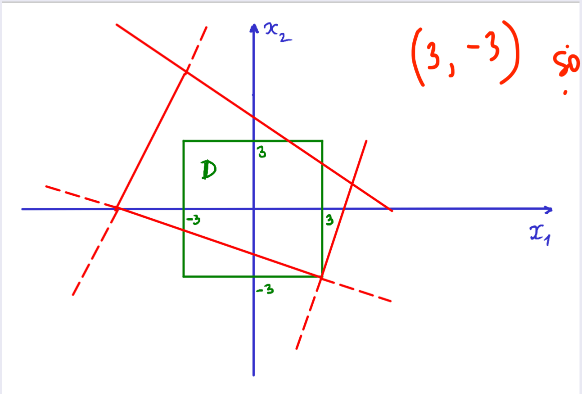
Apply the optimality conditions (gNS) for point $(3, -3)$.

$$\left.\left\{\begin{array}{l}\partial f(y) \cap N(D, y) \neq \{0\} \\ \forall y : \quad f(y) = f(z)\end{array}\right.\right\} \Leftrightarrow y \text{ glob max}$$

Exercice 7

$$f(x_1, x_2) = max\{2x_1 + 3x_2, 3x_1 - x_2, -2x_1 + x_2, -2x_1 - 6x_2\}$$

$$D = \{x \in \mathbb{R}^2 \mid -3 \leq x_i \leq 3, i = 1, 2\}$$



Exercice 8

We consider (CM) in $\mathbb{R}^2$ with

$$f(x_1, x_2) = max\{2x_1 + 3x_2, 3x_1 - x_2, -2x_1 + x_2, -2x_1 - 6x_2\}$$

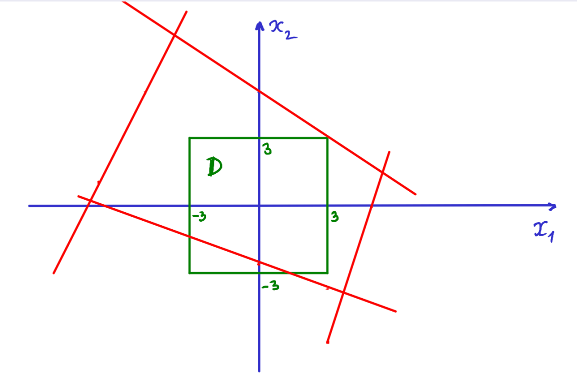
$$D = \{x \in \mathbb{R}^2 \mid -3 \leq x_i \leq 3, i = 1, 2\}$$

Apply the optimality conditions (gNS) for points $(3, 3)$.

Exercice 8

$$f(x_1, x_2) = max\{2x_1 + 3x_2, 3x_1 - x_2, -2x_1 + x_2, -2x_1 - 6x_2\}$$

$$D = \{x \in \mathbb{R}^2 \mid -3 \leq x_i \leq 3, i = 1, 2\}$$



Exercice 9

We consider (CM) in $\mathbb{R}^2$ with

$$f(x_1, x_2) = max\{2x_1 + 3x_2, 3x_1 - x_2, -2x_1 + x_2, -2x_1 - 6x_2\}$$

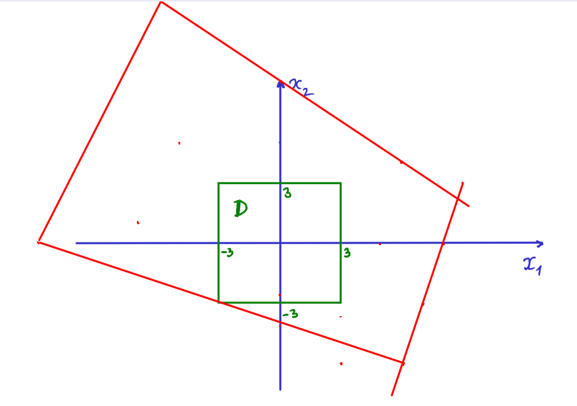
$$D = \{x \in \mathbb{R}^2 \mid -3 \leq x_i \leq 3, i = 1, 2\}$$

Apply the optimality conditions (gNS) for points $(-3, -3)$.

Exercice 9

$$f(x_1, x_2) = max\{2x_1 + 3x_2, 3x_1 - x_2, -2x_1 + x_2, -2x_1 - 6x_2\}$$

$$D = \{x \in \mathbb{R}^2 \mid -3 \leq x_i \leq 3, i = 1, 2\}$$



summary