CIMPA 2021 Mongolia

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Lecture 3

Global Optimality Conditions for (PCMP)

$$\begin{cases} \max F(x) \\ x \in D \end{cases} \qquad (PCMP)$$

D convex compact

$F(x) = \min\{f_1(x), \dots, f_m(x)\}$, $M = \{1, \dots, m\}$

$f_i(x)$ - convex $\forall i = 1, \dots, m$

$m = 1$ $(PCMP) \Longleftrightarrow (CM)$

$$\begin{cases} \max f(x) \\ x \in D \end{cases} \qquad (CM)$$

f convex

- Geometry of GO conditions for (CM)

$$\boxed{\begin{array}{l} z \text{ global maximum of (CM)} \\ \Longleftrightarrow \quad \underset{\text{convex}}{D} \subset \underset{\text{convex}}{\mathcal{L}_f(f(z))} \end{array}}$$

- GO conditions

$$\begin{cases} z \in D \quad \exists v : f(v) < f(z), \quad z \text{ global maximum of (CM)} \\ \Longleftrightarrow \quad \partial f(y) \cap N(D, y) \neq \{0\} \quad \forall y : f(y) = f(z) \end{cases} \qquad (gNS)$$

For z

$I(z) = \{ i \in M \mid f_i(z) = F(z) \}$

set of active functions at z

$D_k(z) = \{ x \in D \mid f_j(x) > F(z) \text{ for all } j \in M \setminus \{k\} \}$

special subdomain

Exercice 10.

$f_1(x) = x_1^2 + (x_2+4)^2 - 36$

$f_2(x) = (x_1+8)^2 + (x_2-3)^2 - 36$

$f_3(x) = x_1^2 + (x_2-8)^2 - 16$

$f_4(x) = (x_1-8)^2 + (x_2-3)^2 - 53$

$F(x) = \min\{ f_1(x), f_2(x), f_3(x), f_4(x) \}$

- Calculer $I(z)$ for $z = (0,5)$
- Draw $D_4(z)$ for $z = (0,5)$

Geometry of GO conditions for (PCMP)

$$\boxed{\begin{array}{l} z \text{ global maximum of (PCMP)} \\ \iff \quad \underset{\text{convex}}{D} \subset \underset{\text{non convex}}{\mathcal{L}_F(F(z))} \end{array}}$$

<u>Lemma 1</u>

If for a point $z \in D$ both

i) $F(z) \geq F(x) \quad \forall x \in D$

ii) $f_k(z) = F(z)$ for some k hold

Then

$$f_k(z) \geq f_k(x) \qquad \forall x \in D_k(z).$$

Proof suppose that $\exists u \in D_k(z)$ such that

$$f_k(u) > f_k(z).$$

$u \in D_k(z) \Rightarrow f_j(u) > F(z) \quad \forall j \in M \setminus \{k\}, \quad u \in D$

$\Rightarrow \quad F(u) > F(z) \geq F(u)$ contradiction.

$$\left\{\begin{array}{l}\text{Lemma 1} \\ (gNS)\end{array}\right. \Rightarrow \text{necessary condition for (PCMP)}$$

<u>Proposition</u>

If $z \in D$ is a global maximum of (PCMP)
then for all $k \in I(z)$

$$\left\{\begin{array}{l}\partial f_k(y) \cap N(D_k(z), y) \neq \{0\} \\ \text{for all } y \text{ such that } f_k(y) = F(z)\end{array}\right. \qquad (gN)$$

Proof:

$$f_k(z) = F(z) \qquad \forall k \in I(z) \subset M$$

$$z \text{ solves (PCMP)} \xrightarrow{\text{Lemma 1}} z \text{ solves } \left\{\begin{array}{l}\max f_k(x) \\ x \in D_k(z)\end{array}\right.$$

$$\forall k \in I(z)$$

$$(gNS) \Rightarrow (gN)$$

Lemma 2

Given vector $c \in \mathbb{R}^n$, $u \in cl\,co(D_k(z))$,
then $\exists\, w \in D_k(z)$ such that $\langle c, u\rangle \le \langle c, w\rangle$.

$\boxed{\text{cl and co stand for closure and convex hull}}$

Lemma 3.

Given continuous functions $g(\cdot)$, $h(\cdot)$,
$\varphi(\cdot) = \min\{g(\cdot), h(\cdot)\}$.
If for all $x \in D$
$h(x) > \varphi(z) \geq g(x)$ for some z,
then $\varphi(z) \geq \varphi(x)$ for all $x \in D$.

Main result (Sufficient Condition)

Theorem

Let $z \in D$ and assume that
$\exists k \in I(z)$ $\exists v$ such that $f_k(v) < f_k(z)$.
Then sufficient condition for z to be a global maximum for (PCMP) is

$$\begin{cases} \partial f_k(y) \cap N(\mathrm{clco}(D_k(z)), y) \neq \{0\} \\ \text{for all } y \text{ such that } f_k(y) = F(z) \end{cases} \qquad (gS).$$

Proof:

$$f_k(z) \geq f(x) \quad \forall x \in \mathrm{clco}(D_k(z))$$

$\Rightarrow$

$$f_k(z) \geq f_k(x) \quad \forall x \in D_k(z). \qquad (1)$$

Denote $\psi_k(x) = \min\{f_j(x) \,/\, j \in M \setminus \{k\}\}$.

$x \in D_k(z) \Rightarrow \psi_k(x) > F(z) \quad \forall x \in D$.

$F(z) = f_k(z)$ since $k \in I(z)$.

$(1) \Leftrightarrow \quad F(z) \geq f_k(x) \quad \forall x \in D \qquad \psi_k(x) > F(z)$

Lemma 3.

$$\Rightarrow F(z) \geq F(x) = \min\{f_k(x), \psi_k(x)\} \quad \forall x \in D. \quad \#$$

Exercice 11. $F(x)$ is from exercice 10.

Draw $\mathcal{L}_F(F(y))$, $y=(6;-4)$.

Check GO condition (gs) for $y=(6;-4)$.

$D=\{x\in\mathbb{R}^2 \,/\, -4\le x_1\le 10,\ -6\le x_2\le 8,\ x_1-x_2\le 10\}$