Introduction to Integer Linear Programming, reformulation and decomposition

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Part 3: decomposition methods

Content

Introduction to duality in linear programming

Illustration

$$\max z = 3x_1 + 2x_2 \quad (1)$$

s.c.

$$2x_1 + x_2 \leq 18 \quad (2)$$

$$2x_1 + 3x_2 \leq 42 \quad (3)$$

$$3x_1 + x_2 \leq 24 \quad (4)$$

$$x_1, x_2 \geq 0 \quad (5)$$

Feasible solutions produce lower bounds for the objective-function value z.

- What if we were interested in upper bounds (UB) ?

Illustration

$$\max z = 3x_1 + 2x_2 \quad (1)$$

s.c.

$$2x_1 + x_2 \leq 18 \quad (2)$$

$$2x_1 + 3x_2 \leq 42 \quad (3)$$

$$3x_1 + x_2 \leq 24 \quad (4)$$

$$x_1, x_2 \geq 0 \quad (5)$$

- Multiply constraint (2) by 2 and deduce an UB of z
- Multiply constraint(3) by $\frac{3}{2}$ and deduce an UB of z
- Consider the linear combination (2)+$\frac{1}{3}$(3) and deduce an UB of z
- Consider the linear combination $\frac{3}{7}$(2)+$\frac{3}{7}$(3)+$\frac{3}{7}$(4) and deduce an UB of z

Illustration

$$\max z = 3x_1 + 2x_2 \quad (1)$$

s.c.

$$2x_1 + x_2 \leq 18 \quad (2)$$
$$2x_1 + 3x_2 \leq 42 \quad (3)$$
$$3x_1 + x_2 \leq 24 \quad (4)$$
$$x_1, x_2 \geq 0 \quad (5)$$

- A linear combination produce an UB if and only if each coefficient of a variable is higher or equal to the coefficient of the same variable in the objective-function
- It is preferable to search for the smallest possible value of UB

$\Downarrow$

Model the problem of finding the best coefficients (linear program)

Exemple

$$\max 3x_1 + 2x_2 \quad (6)$$

s.c.

$$2x_1 + x_2 \leq 18 \quad (7)$$
$$2x_1 + 3x_2 \leq 42 \quad (8)$$
$$3x_1 + x_2 \leq 24 \quad (9)$$
$$x_1, x_2 \geq 0 \quad (10)$$

PRIMAL

$$\min 18u_1 + 42u_2 + 24u_3 \quad (11)$$

s.c.

$$2u_1 + 2u_2 + 3u_3 \geq 3 \quad (12)$$
$$u_1 + 3u_2 + u_3 \geq 2 \quad (13)$$
$$u_1, u_2, u_3 \geq 0 \quad (14)$$

DUAL

1. Solve each of these linear programs
2. Compare the optimal costs
3. Write the matrix form of these two problems and comment on it

Relation Primal / Dual

Duality is a fundamental concept in linear programming. We can associate to a linear program (PL) its dual (DL) defined as a linear program:

- with a matrix of constraints coefficients A^T, with a vector of decision variables u
- a right-hand-side vector c, a cost vector b,

according to the following conversion table.

(LP)	**Primal**	**Dual**	(DL)
$\min z = c.x$	Objective-function (**min**)	Right-hand-side	$\max w = u.b$
	Right-hand-side	Objective-function (**max**)	
s. t.	A constraint matrix	A^T constraint matrix	s. t.
	Constraint i: $\geq$	Variable $u_i \geq 0$	
$A.x \leq b$	Constraint i: $=$	Variable $u_i \in \mathbb{R}$	$u.A^T \leq c$
	Variable $x_j \geq 0$	Constraint j: $\leq$	
$x \geq 0$	Variable $x_j \in \mathbb{R}$	Constraint j : $=$	$u \leq 0$

Table 1: Primal and Dual: rules and examples

(PL) is then called primal and (DL) is its dual problem.

Relation Primal / Dual

There are special connexions between these two problems, including:

- The dual of (DL) is (PL).
- If $\bar{x}$ if a solution of (PL) and $\bar{u}$ is a solution of (DL), then: $\bar{z} = c.\bar{x} \geq \bar{w} = \bar{u}.b$.
- Optimal solutions $x*$ and $u*$ of (PL) and (DL) share the same cost $z* = w*$.
- If solutions of a (PL) and its (DL) share the same cost, then they are optimal
- The reduced cost [1] $\tilde{c}_j$ of the variable $\bar{x}_j$ is:

$$\tilde{c}_j = c_j - \bar{u}A^j \quad (15)$$

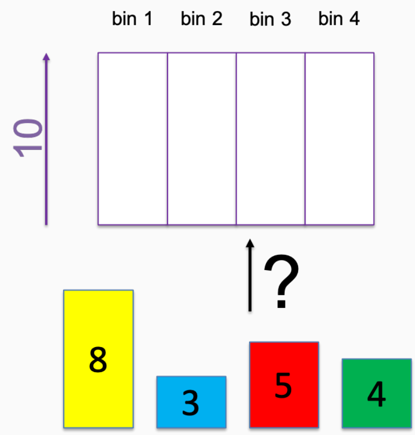
Therefore, in the minimisation case, the cost of a feasible (DL) solution is a lower bound for the optimum, whereas the cost of a feasible (PL) solution is an upper bound.

[1] The reduced cost of a variable corresponds to the impact on the objective-function value of a unitary incrementation of its current value. At the optimum of a minimisation problem, the reduced cost of all its variables is greater or equal to zero because, if there existed variables with negative reduced cost, that would mean that the values of such variables should have been increased to reduce the objective-function value..

Theorem
If either (PL) or (DL) has a finite optimal solution, then so does the other, and the optimum objective values are equal. It either problem has an unbounded objective value, then the other is infeasible.

Introduction to column generation and Branch-and-Price

Our case study: the bin packing problem



Data:

- $|J|$ bins of capacity $Q = 10$,
- $|I| = 4$ items to store,
- item i has a size w_i

How to store items in a minimal number of bins whilst respecting the capacity ?

$$
\begin{aligned}
(CF)\ \min\ & y_1 + y_2 + y_3 + y_4 && (16)\\
s.t.\ & 8x_{1,1} + 3x_{1,2} + 5x_{1,3} + x_{1,4} \leq 10y_1 && (17)\\
& 8x_{2,1} + 3x_{2,2} + 5x_{2,3} + x_{2,4} \leq 10y_2 && (18)\\
& 8x_{3,1} + 3x_{3,2} + 5x_{3,3} + x_{3,4} \leq 10y_3 && (19)\\
& 8x_{4,1} + 3x_{4,2} + 5x_{4,3} + x_{4,4} \leq 10y_4 && (20)\\
& x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4} \in \{0,1\} && (21)\\
& x_{4,1} x_{4,2}, x_{4,3}, x_{4,4}, y_1, y_2, y_3, y_4 \in \{0,1\} && (22)
\end{aligned}
$$

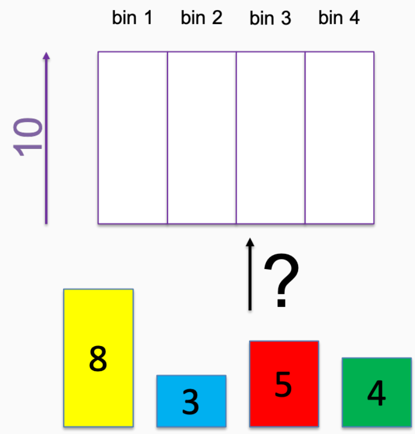
Focus on the linear relaxation of (CF):

- Cost of LP = 2.0

Can we find a formulation of the binpacking whose linear relaxation would provide a better bound?

An extended formulation

Key idea: consider as input data all subsets of items that can be stored together



- set of 1 item: 8, 3, 5, 4
- set of 2 items: (3,5), (3,4), (5,4)
- set of 3 items: -
- set of 4 items: -

Which subset of items should we choose for each bin to reduce number of bins used ?

An extended formulation (EF1)

Minimize the number of subset selected

$$\min \sum_{s=1}^{7} \sum_{j=1}^{4} z_{s,j} \quad (23)$$

s.t.

choose one of the subsets that contain the 1st item

$$z_{1,1} + z_{1,2} + z_{1,3} + z_{1,4} = 1 \quad (24)$$

choose one of the subsets that contain the 2nd item

$$z_{2,1} + z_{2,2} + z_{2,3} + z_{2,4} + z_{5,1} + z_{5,2} + z_{5,3} + z_{5,4} + z_{6,1} + z_{6,2} + z_{6,3} + z_{6,4} = 1 \quad (25)$$

choose one of the subsets that containthe 3rd item

$$z_{3,1} + z_{3,2} + z_{3,3} + z_{3,4} + z_{5,1} + z_{5,2} + z_{5,3} + z_{5,4} + z_{7,1} + z_{7,2} + z_{7,3} + z_{7,4} = 1 \quad (26)$$

choose one of the subsets that contain the 4th item

$$z_{4,1} + z_{4,2} + z_{4,3} + z_{4,4} + z_{6,1} + z_{6,2} + z_{6,3} + z_{6,4} + z_{7,1} + z_{7,2} + z_{7,3} + z_{7,4} = 1 \quad (27)$$

All variables are binary

$$z_{s,j} \in \{0, 1\}, \forall s \in \{1, ..., 7\}, \forall j \in \{1, ..., 4\} \quad (28)$$

No need to know which bin was assigned to which subset ! (EF2)

Minimize the number of subset selected

$$\min z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 \quad (29)$$

s.t.

choose one subset that contains the 1st item (8)

$$z_1 = 1 \quad (30)$$

choose one subset that contains the 2nd item (3)

$$z_2 + z_5 + z_6 = 1 \quad (31)$$

choose one subset that contains the 3rd item (5)

$$z_3 + z_5 + z_7 = 1 \quad (32)$$

choose one subset that contains the 4th item (4)

$$z_4 + z_6 + z_7 = 1 \quad (33)$$

All variables are binary

$$z_s \in \{0, 1\}, \forall s \in \{1, ..., 7\} \quad (34)$$

Quality of the linear relaxation of (EF2)

Let (LEF2) be the linear relaxation of (EF2). Its optimal solution is :

- $z_1 = 1$
- $z_5 = z_6 = z_7 = 0.5$
- $z_2 = z_3 = z_4 = 0$

which corresponds to an objective-function value of 2,5 !

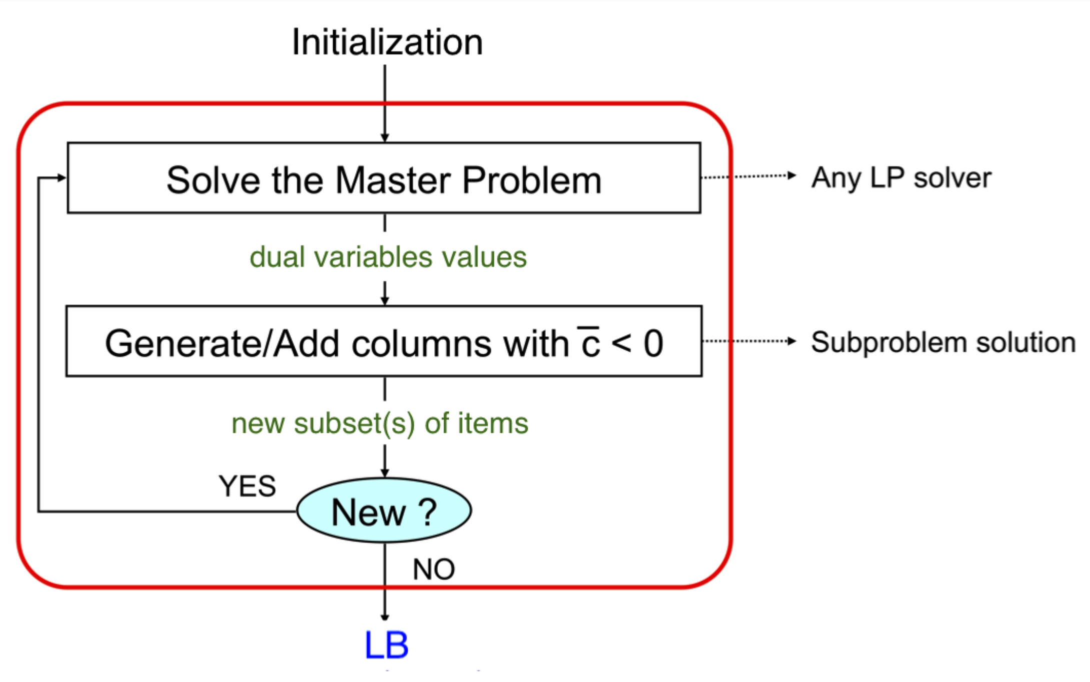
Therefore

- strength of (LEF2): bounds obtained of better quality than (LCF)
- limit of (LEF2): exponential number of variables: for larger instances, it would not be possible to generate (EF2) and feed it to a black-box MILP solver

Key observation: many variables are inactive (set to 0) in the optimal solution of (LEF2).

Column generation

If we could generate only the "good" subset of variables, we could solve (LEF2) without needing to explicitely integrate all the exponential number of variables(=columns)



How to identify promising columns

PRIMAL

$$\min \sum_{s \in \mathcal{S}} z_s \tag{35}$$

$$s.t. \sum_{s \in \mathcal{S}} a_{i,s} z_s = 1, \quad \forall i \in \{\mathcal{I}\} \tag{36}$$

$$z_s \geq 0, \quad \forall s \in \mathcal{S} \tag{37}$$

= master problem

DUAL

$$\max \sum_{i \in \mathcal{I}} u_i \tag{38}$$

$$s.t. \sum_{i \in \mathcal{I}} a_{i,s} u_i \leq 1, \quad \forall s \in \{\mathcal{S}\} \tag{39}$$

$$u_i \in \mathbb{R}, \quad \forall i \in \mathcal{I} \tag{40}$$

How to identify promising columns

PRIMAL

$$\min \sum_{s \in \mathcal{S}} z_s \tag{35}$$

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$$z_s \geq 0, \quad \forall s \in \mathcal{S} \tag{37}$$

= master problem (MP)

DUAL

$$\max \sum_{i \in \mathcal{I}} u_i \tag{38}$$

$$s.t. \sum_{i \in \mathcal{I}} a_{i,s} u_i \leq 1, \quad \forall s \in \{\mathcal{S}\} \tag{39}$$

$$u_i \in \mathbb{R}, \quad \forall i \in \mathcal{I} \tag{40}$$

- A restricted master problem (RMP) is obtained if $\mathcal{S}$ is restricted to a subset $\overline{\mathcal{S}}$
- Any column from $\mathcal{S} \backslash \overline{\mathcal{S}}$ corresponds to a constraint missing from the dual
- A constraint missing from the dual, but respected by the current solution, can be disregarded because adding it to $\overline{\mathcal{S}}$ would not have changed the solution
- given the current (RMP) solution and the resulting dual variable values $\overline{u}_i$, is there a missing subset of items that would lead to a violated constraint (39) ?
  - if yes, some of them should be added to $\overline{\mathcal{S}}$ before (RMP) is solved again
  - if no, stop: the current solution of (RMP) is also optimal for (MP)

Resulting subproblem

Objective:

- Given the dual variables values $\overline{u}_i$ corresponding to your current solution, find the subset of items s^* and therefore the associated coefficients a_{i,s^*} such that $\sum_{i \in \mathcal{I}} a_{i,s^*} \overline{u}_i > 1$.

Finding this set $s*$ consists in solving the following subproblem

$$\max \sum_{i \in \mathcal{I}} \overline{u}_i \alpha_i \quad (41)$$

$$s.t. \sum_{i \in \mathcal{I}} w_i \alpha_i \leq Q \quad (42)$$

$$\alpha_i \in \{0,1\}, \quad \forall i \in \mathcal{I} \quad (43)$$

$\Rightarrow$ a Knapsack problem !

Resulting subproblem

Objective:

- Given the dual variables values $\overline{u}_i$ corresponding to your current solution, find the subset of items s^* and therefore the associated coefficients a_{i,s^*} such that $\sum_{i \in \mathcal{I}} a_{i,s^*} \overline{u}_i > 1$.

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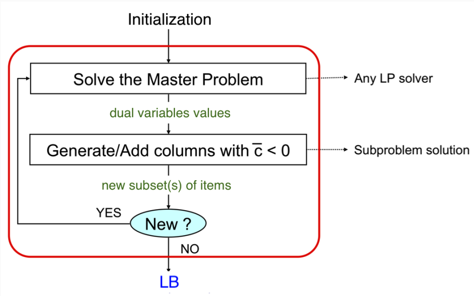
$\Rightarrow$ a Knapsack problem !

The resulting column has a reduced cost $\overline{c} = \sum_{i \in \mathcal{I}} \alpha_i \overline{u}_i - 1$

branch-and-price
||
branch-and-bound
+
column generation



$$\min \sum_{s \in \mathcal{S}} \sum_{j \in \mathcal{J}} z_{s,j} \quad (44)$$

$$\text{s.t.} \sum_{j \in \mathcal{J}} x_{i,j} = 1, \quad \forall i \in \mathcal{I} \quad (45)$$

$$x_{i,j} = \sum_{s \in \mathcal{S}} a_{i,s} z_{s,j}, \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J} \quad (46)$$

$$z_{s,j} \geq 0, \quad \forall s \in \mathcal{S}, \forall j \in \mathcal{J} \quad (47)$$

$$x_{i,j} \in \{0,1\}, \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J} \quad (48)$$

Conclusion and useful references

Reminder of the goals of this course

We planned to

- introduce you to the basics of (Mixed) Integer Linear Programming (part 1)
- illustrate how MILP can be used to solve Mixed Integer Nonlinear Problems via piecewise linear approximation (part 2)
- introduce you to principles of the column generation and branch-and-price algorithms (part 3)

With the aim of

- expanding your modeling and solving capabilities

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phpsimplex (educational tool to solve a linear program with simplex or the graphical method): http://www.phpsimplex.com/en/

Thank you for your attention !

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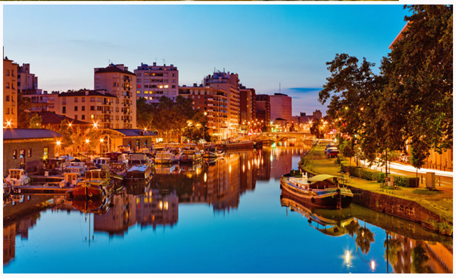


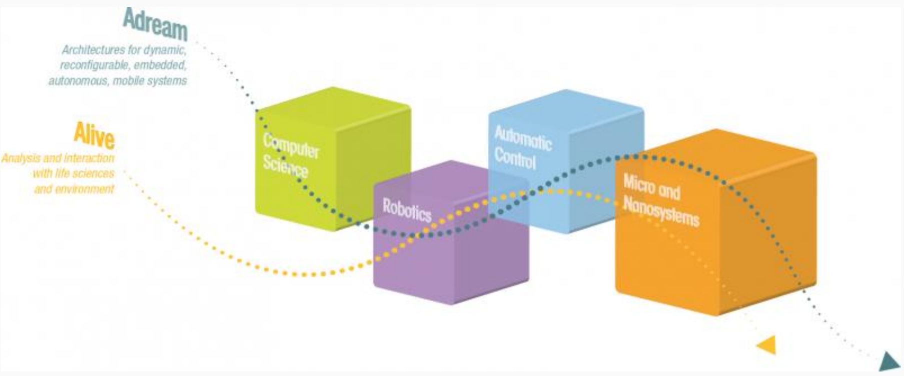
4th largest city in France
2nd largest number of students
2019,2020 favorite student city

https://www.inp-toulouse.fr/en

LAAS-CNRS

Largest lab. of CNRS (French National Center for Scientific Research)

- 642 employees: 43% PhD+PostDoc, 37% Prof.+Researchers
- 4 Disciplines / 8 Scientific Themes / 22 Research Teams



- Critical Information Processing, Networks and Communications, Robotics, Decision and Optimization, MicroNanosystems RF and Optical, Energy Management, Nano-Engineering and Integration, MicroNanoBio Technologies

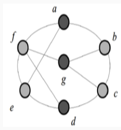
https://www.laas.fr/public/en

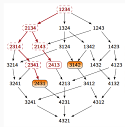
ROC Research Team

(Operations Research, Combinatorial Optimization and Constraints)

- branches of Operations Research and/or Artificial Intelligence (more specifically Constraint Programming).
- *Find structural properties and propose solution approaches for combinatorial optimization problems*
- Research Topics

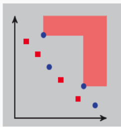
Combinatorial Optimization

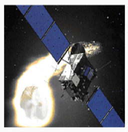




Solution approaches models and algorithms

Robust, Multi-agent, Multi-objective Problems





Industrial applications

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