Optimization Applications in Economics and Finance

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Introduction

Historical development

- Isaac Newton (1642-1727)
  (The development of differential calculus
  methods of optimization)

- Joseph-Louis Lagrange (1736-1813)
  (Calculus of variations, minimization of functionals,
  method of optimization for constrained problems)

- Augustin-Louis Cauchy (1789-1857)
  (Solution by direct substitution, steepest
  descent method for unconstrained optimization)

Historical development

- Leonhard Euler (1707-1783)
  (Calculus of variations, minimization of
  functionals)

- Gottfried Leibniz (1646-1716)
  (Differential calculus methods
  of optimization)

İsim: Gottfired Wilhelm von Leibniz

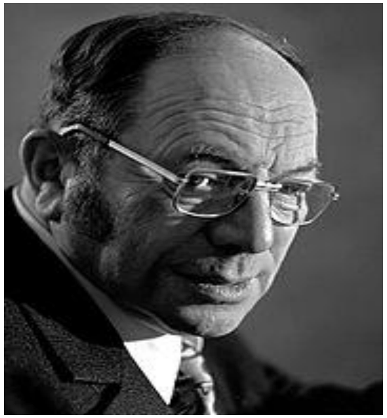
Historical development

- George Bernard Dantzig (1914-2005)
  (Linear programming and Simplex method (1947))

- Richard Bellman (1920-1984)
  (Principle of optimality in dynamic
  programming problems)

- Harold William Kuhn (1925-)
  (Necessary and sufficient conditions for the optimal solution of
  programming problems, game theory)

Leonid Vitaliyevich Kantorovich(1912-1986, Soviet mathematician and economist, Nobel Prize, 1975)



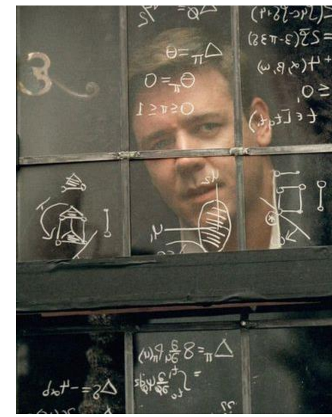
He is the founder of Linear Programming (1939):

$$\min\langle c,x\rangle$$

subject to: $Ax=b$

Historical development

- Albert William Tucker (1905-1995)
  (Necessary and sufficient conditions
  for the optimal solution of programming
  problems, nonlinear programming, game
  theory: his PhD student
  was John Nash)



- Von Neumann (1903-1957)
  (game theory)

Von Neumann proved his minimax theorem in 1928.
Theory of Games and Economic Behavior,
with Morgenstern, O., Princeton Univ. Press, 1944

Lev Semyonovich Pontryagin (1908–1988),

He is the founder of Optimal Control Theory

His maximum principle is fundamental to the modern theory of optimization.

Optimization in Economics

Pietro Verri (Italian Economist)

“Reflications on Polytical Economy”, 1771.

Frisi’s commentary:

$$P = \frac{C}{N}$$

$$dP = 0$$

C: number of buyers,

N: Number of sellers.

Georg von Buquoy(German economist, 1815)

Profit Maximization (in Robertson, Mathematical Economics before Cournot,1949)

$$MR = MC,$$

MR: marginal revenue,

MC: marginal cost.

Agustin Cournot (French economist,1838)

- He analysed profit maximization under perfect competition and monopoly.

$$\max P(x,y), \quad x=\text{const},$$

$$\max Q(x,y), \quad y=\text{const},$$

P,Q are profit functions of two firms(duopoly).

It is due to A.Cournot that optimization became of integral part of economic analysis.

- Herman H.Gossen (German economist,1854) formulated first-order condition for utility maximization problem(UMP) under linear budget constraint. (in Gossen,H.H, The laws of human relations and the rules of human

action derived therefrom, trans. Cambridge: MIT press, 1983)

$$\max u(x), \text{ subject to } \langle p,x \rangle = m.$$

- Leon Walras (French mathematical economist, 1874) formulated first and second order conditions for UMP.

He is founder of modern equilibrium theory

$$S(p)=D(p),$$

S: supply,

D:demand,

p: price

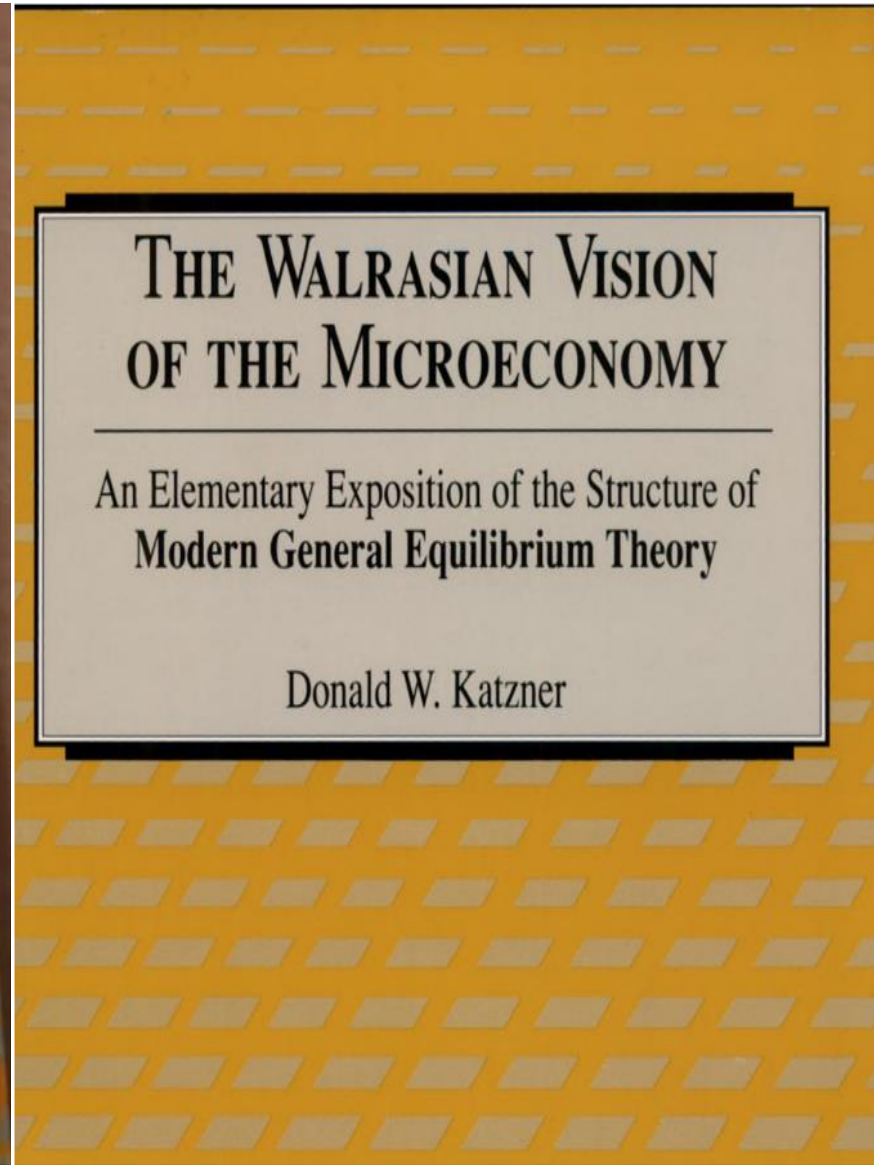
L.Walras, “Elements of Pure Economics”, Switzerland,1874.

- **Kenneth Joseph Arrow** (1921 – 2017, Nobel prize, 1972)

General Equilibrium Theory:

There exists a equilibrium price p such that:

$$S(p)=D(p).$$



2002 (translated by R. Enkhbat)

- "…. the model of constrained utility maximization can not be used successfully

to explain consumer behavior in Japan"

in Katzner, D.W. Time, Ignorance, and Uncertainty in Economic Models, University of Michigan Press, 1988.

Optimization is used in

1. Microeconomics (behavior of individual consumers and firms)
2. Macroeconomics (welfare maximization, optimal taxation, optimal economic growth)
3. Economic theory and analysis

Optimization based Economic Theories

- **Harry Max Markowitz** (born 1927, Nobel prize, 1990)

Markowitz, H.M. "Portfolio Selection". *The Journal of Finance*. **7** (1): 77–91, 1952.

He is the founder of modern portfolio theory.

Markowitz problem 1 (Minimum-variance problem)

$$\min \sigma^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j \,,$$

subject to

$$\sum_{i=1}^{n} x_i = 1,$$

$\sigma^2 : variance,\ \sigma_{ij} : covariance, x_i : weight$

Markowitz problem 2 (Variance and Return)

$$\min \sigma^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \, ,$$

subject to

$$\sum_{i=1}^{n} r_i x_i \geq r_0$$

$$\sum_{i=1}^{n} x_i = 1,$$

$r_i : return$

Markowitz problem 3 (Maximum growth rate)

$$max \sum_{i=1}^{n} r_i x_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \,,$$

subject to

$$\sum_{i=1}^{n} x_i = 1,$$

Markowitz problem 4 (Return maximization)

$$max \sum_{i=1}^{n} r_i x_i ,$$

subject to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \leq \sigma_0^2 ,$$

$$\sum_{i=1}^{n} x_i = 1,$$

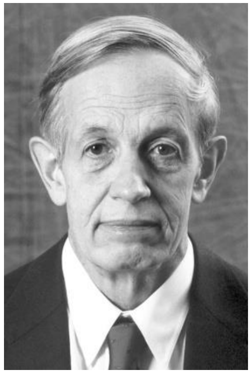
Markowitz problem 5 (Variational calculus formulation)

$$\max \mathcal{J} = \int_0^T \langle r(t), x(t) \rangle e^{-\rho t} dt ,$$

subject to

$$x^T(t) C x(t) \leq \sigma^2(t),$$
$$\sum_{i=1}^{n} x_i(t) = 1,$$
$$t \in [0, T].$$
$$x(t) = (x_1, x_2, \dots, x_n)$$
$$r(t) = (r_1, r_2, \dots, r_n)$$
$$C = \{\sigma \dots\}$$

John Forbes Nash Jr. (1928 – 2015, Nobel Prize, 1994)



He is the founder of non-cooperative game theory.

Max $f_1=p(x+y)x-C_1(x)$,

Max $f_2=p(x+y)y-C_2(y)$,

(x^*,y^*) is Nash equilibrium:

$f_1(x^*,y^*)=\max f_1(x,y^*)$,

$f_2(x^*,y^*)=\max f_2(x^*,y)$,

f_1, f_2 : profit functions,
x,y :quantities,
p: price.

Robert Merton Solow (born 1924, Nobel Prize, 1987)

$$\max c$$

Subject to

$$\frac{dk}{dt} = 0$$

Y=$\alpha^{*}Y + (1-\alpha^{*})$Y,
I=$\alpha^{*}Y$
C= $(1-\alpha^{*})$Y

c: *consumption per capita*
k: *capital per labor*
Y: *output*
α^{*}: *golden rule of accumulation*
I: investment
C: consumption