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Question

When the level set $S = \{x \mid f(x) \leq f(x_0)\}$ is bounded, the Steepest Descent method converges to the set X^{**} of critical points of f. We define the following error term :

$$\epsilon_f(x) = \mid \nabla f(x) \mid^2,$$

The set X^{**} towards which the trajectory converges is exactly the set where $\epsilon_f(.) = 0$, so that $\epsilon_f(.)$ indeed can be viewed as something which measures the "residual in the inclusion $x \in X^{**}$". We assume that the objective function f is continuously differentiable with Lipschitz continuous gradient:

$$\mid \nabla f(x) - \nabla f(y) \mid \leq L_f \mid x - y \mid, \forall x, y \in \Re^n.$$

i Show that

$$f(y) \leq f(x) + (y-x)^T \nabla f(x) + \frac{L_f}{2} \mid y - x \mid^2, \forall x, y \in \Re^n.$$

(*Hint: let* $\phi(\gamma) = f(x + \gamma(y-x))$ *and show that*

$$\mid \phi^{'}(\alpha) - \phi^{'}(\beta) \mid \leq L_f \mid \alpha - \beta \mid^2 \mid \alpha - \beta \mid, \forall \alpha, \beta \in \Re^n.$$

ii Hence, by considering the construction of the Steepest Descent given as:

$$f(x_t) = \min_{\gamma \geq 0} f(x_{t-1} - \gamma \nabla f(x_{t-1})),$$

show that

$$f(x_{t-1}) - f(x_t) \geq \frac{1}{2L_f} \mid \nabla f(x_{t-1}) \mid^2$$

and deduce that

$$\epsilon_f(t) \equiv \min_{0 \leq t < N} \mid \nabla f(x_t) \mid^2 \leq \frac{\eta L_f}{2\epsilon(1-\epsilon)N} \left(f(x_0) - \min f \right).$$

End-Of-Question