ECP 2ème année
Semestre 4, Electif 9, Année 2012
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NUMERICAL OPTIMIZATION AND APPLICATIONS

2012, March 30th, 3 hours

All Scilab programs must be sent at the adress `laurent.dumas@uvsq.fr` *at the end of the exam with a file name of the type* `ECP2012-firstname-name.sci` *and with the subject ECP2012.*

The problem is in english but candidates can give a copy in french if they prefer.

A. Theoretical part

Exercice 1 – Solve the following linear program using the simplex method

$$\begin{array}{l} \max 5x_1 - 2x_2 + x_3, \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0, \\ x_1 - x_2 + x_3 \leq 1, \\ 2x_1 + 2x_2 + x_3 \leq 10, \\ 4x_1 + x_2 + 5x_3 \leq 12. \end{array}$$

Exercice 2 – Consider the optimization problem

$$(\mathscr{P}) \quad \begin{cases} \min f(x,y),\ (x,y) \in \mathbb{R}^2, \quad \text{with} \\ g(x,y) \geq 0, \\ y \geq 0. \end{cases}$$

where

$$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2,$$

$$g(x,y) = \frac{1}{2}(x+y)^2 + (x-y)^2 - 4.$$

1. Explain why the problem $(\mathscr{P})$ has at least one solution.
2. Write the first order KKT optimality conditions for the problem $(\mathscr{P})$.
3. Find all the solutions of $(\mathscr{P})$.

Exercice 3 –

1. When the level set $S = \{x \mid f(x) \leq f(x_0)\}$ is bounded, the Steepest Descent method converges to the set X^{**} of critical points of f. We define the following error term :
$$\epsilon_f(x) = \mid \nabla f(x) \mid^2,$$
The set X^{**} towards which the trajectory converges is exactly the set where $\epsilon_f(.) = 0$, so that $\epsilon_f(.)$ indeed can be viewed as something which measures the "residual in the inclusion $x \in X^{**}$". We assume that the objective function f is continuously differentiable with Lipschitz continuous gradient :
$$\mid \nabla f(x) - \nabla f(y) \mid \leq L_f \mid x - y \mid, \forall x, y \in \Re^n.$$
(i) Show that
$$f(y) \leq f(x) + (y-x)^T \nabla f(x) + \frac{L_f}{2} \mid y - x \mid^2, \forall x, y \in \Re^n.$$
(ii) Hence, by considering the construction of the Steepest Descent given as :
$$f(x_t) = \min_{\gamma \geq 0} f(x_{t-1} - \gamma \nabla f(x_{t-1})),$$
and its trajectory $\{x_t\}$ started at x_0, show that
$$f(x_{t-1}) - f(x_t) \geq \frac{1}{2L_f} \mid \nabla f(x_{t-1}) \mid^2$$
and deduce that for any integer N,
$$\epsilon_f[t] \equiv \min_{0 \leq t < N} \mid \nabla f(x_t) \mid^2 \leq \frac{2L_f}{N} \left(f(x_0) - \min f \right).$$

2. Consider applying the method of steepest descent with exact line searches to the problem
$$\min_x x^T H x,$$
where H is a positive definite Hessian.
(i) Show that $x_* = 0$ solve the above problem.
(ii) A Line-search in direction p from x gives
$$f(x + \gamma p) = \frac{1}{2}(x + \gamma p)^T H (x + \gamma p).$$

Using the exact line-search condition $\frac{df}{dx} = 0$, show that the steplength γ obtained by performing an exact line-search from x in the direction p is given by

$$\gamma = \frac{-p^T g}{p^T H p},$$

where g is the gradient at x.

(iii) Let H be a diagonal matrix given by

$$H = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

(iv) If the starting point $x_1 = \left(\frac{\sigma}{\lambda_1}, 0, \cdots, 0, \frac{1}{\lambda_n}\right)^T$ is chosen, where $\sigma = \pm 1$, show that

$$x_2 = \frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\left(-\frac{\sigma}{\lambda_1}, 0, \cdots, 0, \frac{1}{\lambda_n}\right)^T.$$

(v) Hence, show that at iteration $k+1$ the iterate is

$$x_{k+1} = \left(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\right)^k \left((-1)^k \frac{\sigma}{\lambda_1}, 0, \cdots, 0, \frac{1}{\lambda_n}\right)^T.$$

(vi) What can you say about the speed of convergence, if $\lambda_1 >> \lambda_n$?

B. Computational part

Exercice 4 – We want to find the minimum of the following function :

$$\mathbf{f}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \begin{pmatrix} 20 & 5 \\ 5 & 2 \end{pmatrix} \mathbf{x} - [14\ 6]\mathbf{x} + 10.$$

Write a SCILAB/ MATLAB script to implement the steepest descent algorithm for the above problem as follows :

1. Sketch a contour plot of this function.

2. Using an initial guess $\mathbf{x_0}^T = [40 \; -100]^T$, a suitable learning rate α and a tolerance $\epsilon = 1.0e-09$, generate the iterates using the steepest descent method. Sketch the trajectory of the steepest descent algorithm on the contour plot of part (i).
3. What is the maximum stable learning rate ?

Exercice 5 – Differential evolution (DE) is an evolutionary algorithm (like GA, ES, PSO) seeking for a global minimum of a real valued cost function $J : \mathbb{R}^n \to \mathbb{R}$. DE makes evolve a population of N_{pop} elements (or individuals). The algorithm can be described as follows (where $CR \in [0,1]$ and $F \in [0,2]$ are two parameters described below) :

(i) Random initialization of a population of N_{pop} elements

(ii) From generation 1 to generation N_{gen}, do :

(iii) For each individual $x \in \mathbb{R}^n$:

- Pick randomly three individuals a, b and $c \in \mathbb{R}^n$, distinct from each others and distincts from x.
- Pick i_0 a random index in $\{1, ..., n\}$ and compute the element $y = (y_1, ..., y_n)$ as follows :

  $$\forall i \in \{1, ..., n\}, \quad y_i = a_i + F(b_i - c_i) \text{ if } (r_i < CR) \text{ or } (i = i_0), \text{ else } y_i = x_i$$

  where r_i is randomly chosen in $[0, 1]$.
- If $J(y) < J(x)$, replace x by y in the population

(iv) End generation

1. Write a SCILAB/MATLAB script that is seeking for the minimum of a given function $J : [a,b]^n \to \mathbb{R}$ with a differential evolution of parameters N_{pop}, N_{gen}, CR and F.
2. Apply the DE script to find the minimum of the Rastrigin function $J : [-5,5]^n \to \mathbb{R}$ where

   $$J(x_1, ..., x_n) = \sum_{i=1}^{n} (x_i^2 - \cos(2\pi x_i)) + n$$

   with the parameters $N_{pop} = 10n$, $F = 0.8$ and $CR = 0.2$.