ECP 2ème année
Semestre 4, Electif 9, Année 2013
Mr DUMAS, DAUHOO

NUMERICAL OPTIMIZATION AND APPLICATIONS

2013, April 5th, 3 hours

A l'attention des surveillants : première 1h30 : pas de documents autorisés, deuxième 1h30 : documents et ordinateur avec internet autorisé.

All Scilab programs must be sent at the adress `laurent.dumas@uvsq.fr` *at the end of the exam with a file name of the type* `ECP2013-firstname-name.sci` *and with the subject ECP2013.*

The problem is in english but candidates can give a copy in french if they prefer.

A. Theoretical part

Exercice 1 –

Solve the following linear program using the simplex method

$$\begin{array}{l} \max x_1 + 2x_2 + x_3, \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0, \\ x_1 + 4x_2 + 2x_3 \leq 10, \\ x_1 + x_3 \leq 2, \end{array}$$

Exercice 2 –

We consider the minimization of

$$J(x_1, \cdots, x_n) \ = \ -\sum_{i=1}^{n} \ln(x_i + \alpha_i),$$

subject to

$$x_i \geq 0, \forall i = 1, \cdots, n, \sum_{i=1}^{n} x_i \ = \ 1,$$

where $\alpha \ = \ (\alpha_1, \cdots, \alpha_n)^T \in \mathbb{R}^n$ such that $\alpha_i > 0, \forall i = 1, \cdots, n$.

(i) Show that there exists a solution $u = (u_1, \cdots, u_n)^T$ to the above problem.

(ii) Show that this solution is unique.

(iii) Derive the Karush-Kuhn-Tucker relations associated to the above problem.

(iv) Hence, show that there exists $\xi \in \mathbb{R}$ such that $u_i = \max(0, \xi - \alpha_i)$, for $i = 1, \cdots, n$.

Exercice 3 –

In order to minimize the function $f(x) = \|x\|^3$, where $x \in \mathbb{R}^n$ and $\|x\| = \left(x_1^2 + \cdots + x_n^2\right)^{\frac{1}{2}}$, the Newton method given by :

$$x^{(k+1)} = x^{(k)} - \left(\nabla^2 f\left(x^{(k)}\right)\right)^{-1} \nabla f\left(x^{(k)}\right),$$

is used iteratively.

(i) Find $\nabla f(x)$ and $\nabla^2 f(x)$ respectively.

(ii) Using the following formula,

$$\left(I + \frac{1}{2}uu^T\right)^{-1} = I - \frac{1}{2}uu^T,$$

where I is the identity matrix of order n and $u \in \mathbb{R}^n$, find $\left(\nabla^2 f(x)\right)^{-1}$ and hence, write down the Newton iterative formula.

(iii) Show that the Newton iteration formula converges **linearly** to the optimal solution $x^* = 0$.

B Computational part

Exercice 4 –

Write a SCILAB/ MATLAB script to implement the simplex algorithm in its canonical form : min $J(x)$ with the constraints $Ax \leq b$ and $x \geq 0$ **by using the method presented in the course** based on successive arrays.

Apply this algorithm to verify the solution obtained in Exercice 1.

Exercice 5 –

We want to solve the following problem :

$$\min x_1^2 + x_2^2,$$

subject to

$$\begin{array}{rcl} x_1 - x_2 + 1 & \geq & 0 \\ 2x_1 + x_2 - 2 & \geq & 0. \end{array}$$

Write a SCILAB/ MATLAB script to implement the Uzawa algorithm for the above problem.

Exercice 6 –

Particle Swarm Optimization is an optimization method based on the evolution of a swarm, that has been presented in the last computational session.

Starting from the code written during this session and available at the following adress :

`http://dumas.perso.math.cnrs.fr/ECP2011-PSO.sci`

write a SCILAB/MATLAB graphical interface that shows the evolution of the convergence of the swarm to the global optimum of the Rastrigin function.