Short Course: Multidisciplinary optimization and industrial applications 2023

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&

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Day 1 Wednesday 11/01/2023

Venue: RBLT

Session 1: 13.00-14.30 (**MZD & LD**)

Introduction, motivation, Notion of optimization with or without constraints

(1)

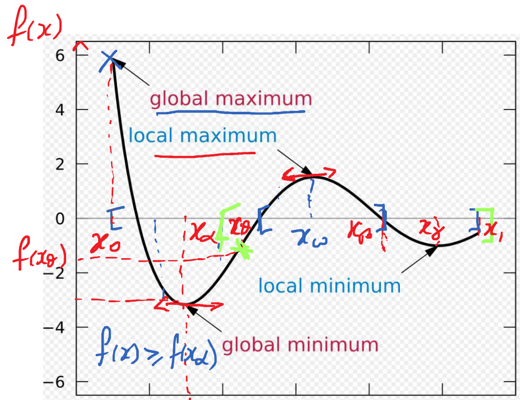
optimisation

finding the maximum or the minimum of a given function $f(x)$

$f(x_1, x_2, x_3)$

differentiable functions.

$\min f(x)$; $\max_{x \in S} g(x) \rightarrow \min_{x \in S} -g(x)$



on the interval $[x_0, x_1]$,

$$\min_{x \in [x_0, x_1]} f(x) = f(x_\alpha) \simeq -3$$

for all $x \in [x_\beta, x_1]$, $f(x) \geq f(x_\gamma)$

however when $x \notin [x_\beta, x_1]$,

$f(x)$ is not necessarily $\geq f(x_\gamma)$

ex: $f(x_\theta) < f(x_\gamma)$

(2) a local minimum of $f(x)$ occurs at x_γ provided there exists an interval I such that $x_\gamma \in I$ and for all $x \in I$,

$$f(x) \geq f(x_\gamma)$$

(3) Can a global min (max) be a local min (max)?

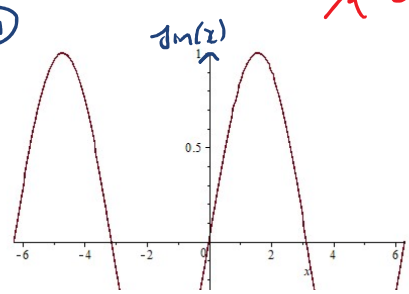
YES!

A local min is not necessarily a global min.

global min $\Rightarrow$ local min

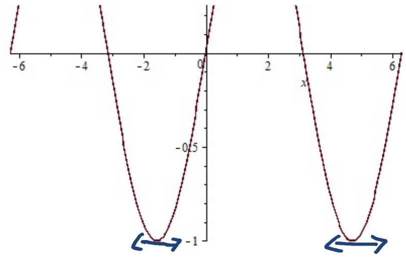
local min $\not\Rightarrow$ global min.

(4)



Can we have more than 1 global minimum (resp. max.)

Yes, however the value of $f(x)$ at each of the global minima is the same.



each of the global minima is the same.

for all $x \in [-2\pi, 2\pi]$,

$\sin(x) \geq -1 = \sin(-\pi/2) = \sin(3\pi/2)$

$f(x) \geq f(-\pi/2) = f(3\pi/2)$, for all $x \in [-2\pi, 2\pi]$

$f(x)$ has 2 global minima on $[-2\pi, 2\pi]$

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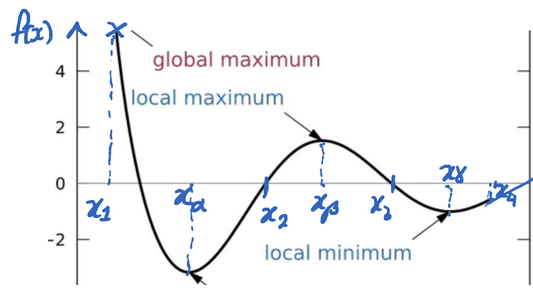
(5)

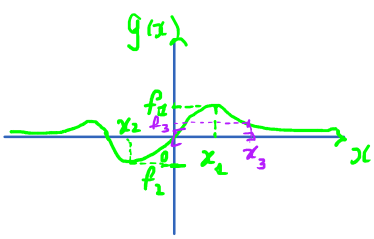


$$\min_{x \in [x_1, x_4]} f(x) = f(x_\alpha)$$

$$\max_{x \in [x_1, x_4]} f(x) = f(x_1)$$

$$\max_{x \in [x_2, x_3]} f(x) = f(x_\beta)$$

$$\max_{x \in \mathbb{R}} f(x) = +\infty$$

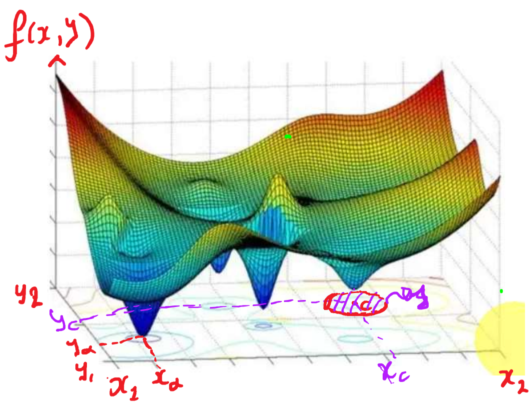


$$\max_{x \in \mathbb{R}} f(x) = f_1 = f(x_1)$$ unconstrained optimisation problem

$$\max_{x \geq x_3} f(x) = f_3 = f(x_3)$$

$$\max_{x \in [0, x_3]} f(x) = f_1 = f(x_1)$$ constrained optimisation problem

(6)



$$\min_{(x,y) \in \mathbb{R}^2} f(x,y) = f(x_\alpha, y_\alpha)$$ Unconstrained optimisation problem

$$\min_{(x,y) \in S} f(x,y) = f(x_c, y_c)$$ Constrained optimisation problem

$$\min_{\sqrt{(x-x_c)^2+(y-y_c)^2} \leq r} f(x,y) = f(x_c, y_c)$$

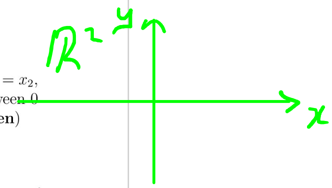
1 Convex Sets

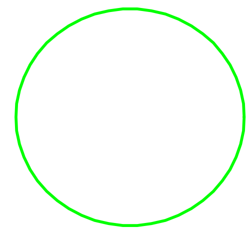
1.1 Lines and line segments

Suppose $x_1 \neq x_2$ are two points in $\mathbb{R}^n$. Then, points of the form

$$y = \lambda x_1 + (1-\lambda)x_2 \quad \text{where} \quad \lambda \in \mathbb{R},$$

form the *line* passing through x_1 and x_2. The parameter value $\lambda = 0$ corresponds to $y = x_2$, and the parameter value $\lambda = 1$ corresponds to $y = x_1$. Values of the parameter λ between 0 and 1 correspond to the (closed) line segment between x_1 and x_2. (**Boyd and Vanden**)





1.2 Convex Sets

Definition 1.1 *Let C be a given set, $C \subset \mathbb{R}^2$, $(x,y) \in C$ and $(x',y') \in C$. C is convex if and only if*

$$\lambda(x,y) + (1-\lambda)(x',y') \in C \quad \textit{for} \quad 0 < \lambda < 1.$$

For any $\alpha \in \ell$,

$\alpha\,(x_\alpha, y_\alpha)$

there exists $0 \le \lambda \le 1$ such that

$(x_\alpha, y_\alpha) = \lambda(x,y) + (1-\lambda)(x',y')$

$(x_\beta, y_\beta) = \lambda_1(x,y) + (1-\lambda_1)(x',y')$

Convex linear combination

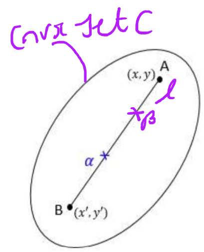


Figure 1: A convex set

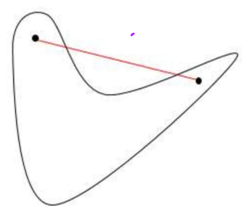
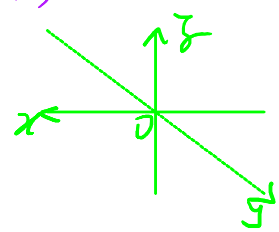


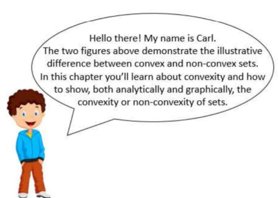
Figure 2: A non-convex set

The point $\alpha = \lambda(x,y) + (1-\lambda)(x',y')$ is said to be a *linear combination* of points A and B as shown in Figure 1.

$\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, ..., $\mathbb{R}^n$

Convex sets





2 Convex Function

Definition 2.1 *A function $f(x, y)$ is said to be **convex** if and only if*

$$f(\lambda(x, y) + (1 - \lambda)(x', y')) \leq \lambda f(x, y) + (1 - \lambda) f(x', y') \quad \text{for} \quad 0 < \lambda < 1.$$

Definition 2.2 *A function $f(x, y)$ is said to be **strictly convex** if and only if*

$$f(\lambda(x, y) + (1 - \lambda)(x', y')) < \lambda f(x, y) + (1 - \lambda) f(x', y') \quad \text{for} \quad 0 < \lambda < 1.$$

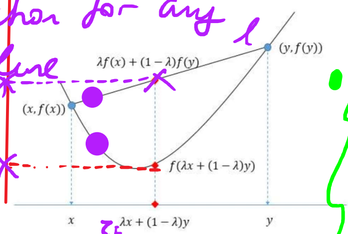


Figure 8: A convex function

For a concave function,

$$f(\lambda(x, y) + (1 - \lambda)(x', y')) \geq \lambda f(x, y) + (1 - \lambda) f(x', y') \quad \text{for} \quad 0 < \lambda < 1.$$

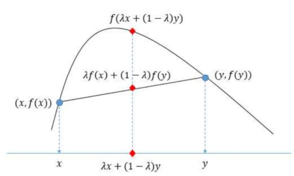


Figure 9: A concave function

2.2 Hessian

Let $f(x_1, x_2, x_3) = x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2$.
The Hessian matrix related to $f(x_1, x_2, x_3)$ is given by

$$\nabla^2 f(x_1, x_2, x_3) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{pmatrix}$$

Definition 2.7 *Let f be a function of two or more variables with continuous first order and second order derivatives on an open and convex set S. let the Hessian of f be denoted by $H(X)$. Then*

(i) f is concave if and only if $H(X)$ is negative semidefinite for $X \in S$.

(ii) If $H(X)$ is negative definite for $X \in S$, then f is strictly concave.

(iii) f is convex if and only if $H(X)$ is positive semidefinite for $X \in S$.

$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0$

(iv) f is strictly convex if and only if $H(X)$ is positive definite for $X \in S$.

$\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$

Definition

A set $D \subset R^2$ is:

- bounded if it can be enclosed in a circle;

- closed if it includes its boundary (denoted by ∂D);

We say D is compact if it is both closed and bounded.

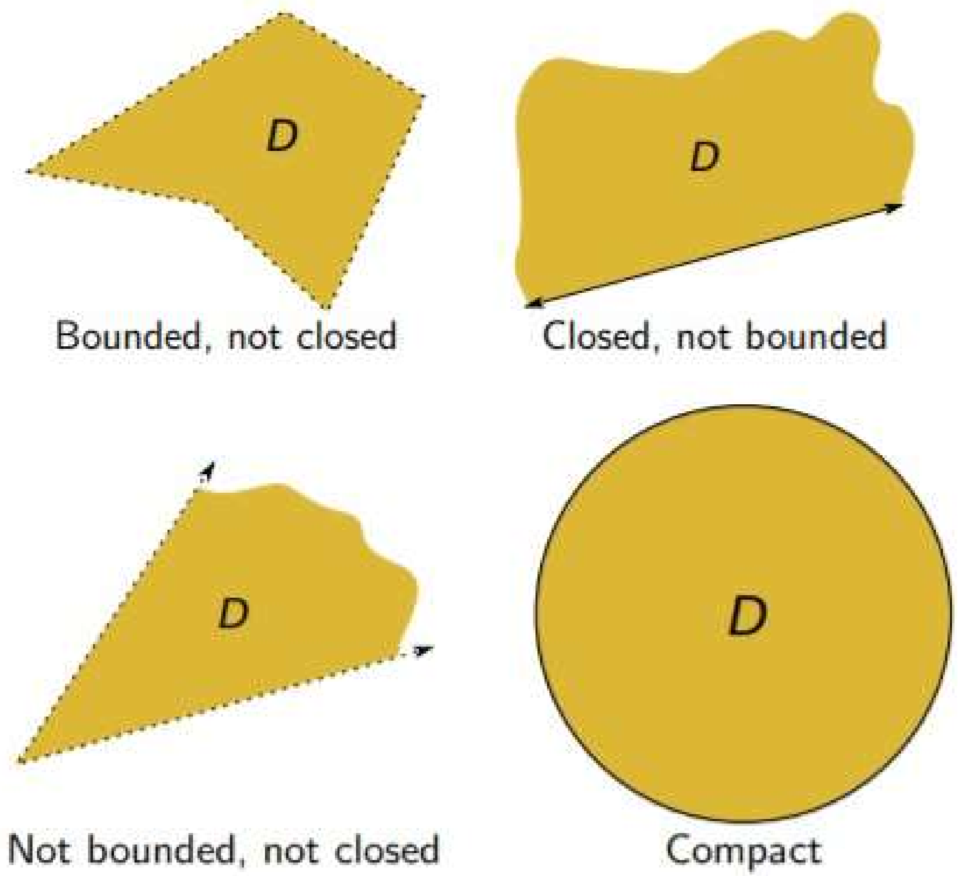


Figure 1: Closedness, boundedness and compactness

A Few Useful Theorems

Theorem 1

If $f(\mathbf{x})$ is continuous on a compact set $D \subset \mathbb{R}^n$, then f has global extrema there.

Theorem 2

If $f(\mathbf{x})$ is differentiable on a compact set $D \subset \mathbb{R}^n$, then the extrema of f must occur:

Either

- At critical points within D;

OR

- On the boundary ∂D.

Theorem 3

Let $f(\mathbf{x})$ be a continuous function defined on all of $\mathbb{R}^n$.
If $f(\mathbf{x})$ is coercive, then $f(\mathbf{x})$ has a global minimizer.
Furthermore, if the first partial derivatives of $f(\mathbf{x})$ exist on all of $\mathbb{R}^n$,
then any global minimizers of $f(\mathbf{x})$ can be found among the critical points of $f(\mathbf{x})$.

Optimisation Taylor Series 01/01/2023

1. **Second order Taylor series based expansion of a multivariable function**
   Let $f : D \subseteq R^n \to R$, be of class C^2 on the open interval D. Let $x^*, x \in D$ such that $[x^*, x] \subset D$ and there exists $z \in x^*,\, x$ such that

$$f(x) = f\left(x^*\right) + \nabla f\left(x^*\right) \cdot \left(x - x^*\right) + \frac{1}{2}\left(x - x^*\right) \cdot \nabla^2 f(z)\left(x - x^*\right)$$

   where

$$\nabla f(.) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(.) \\ \vdots \\ \frac{\partial f}{\partial x_n}(.) \end{pmatrix}, \nabla^2 f(.) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(.) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(.) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(.) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(.) \end{pmatrix}$$

2. **Alternative formulation of the second order Taylor series based expansion of a multivariable function**
   For all $x \in V_r\left(x^*\right) \subseteq D$,

$$f(x) = f\left(x^*\right) + \nabla f\left(x^*\right) \bullet \left(x - x^*\right) + \frac{1}{2}\left(x - x^*\right) \cdot Hf\left(x^*\right)\left(x - x^*\right) + \left\|x - x^*\right\|^2 \varepsilon\left(x - x^*\right),$$

   where ε is a function such that $\lim_{h \to 0} \varepsilon(h) = 0$

Lecture Example 08/06/2022
Consider the funchon $f : \mathbb{R}^2 \to \mathbb{R}$ and defined as fotlows:

$$f(x, y) = x^4 + y^4 - 4xy \quad 4xy.$$

and the set:

$$C = \left\{(x, y) : |x| \leq \sqrt{2}, |y| \leq \sqrt{2}\right\}$$

(i) Show that the probtem given by

$$\min_{(x,y)\in C} f(x, y)$$

has at least one solution.

(ii) Find the critical point(y) of $f(x, y)$

(iii) Determine the mimimum point(s) of $f(x, y)$ on $\mathbb{R}^2$, stating whether there are local or global minimum points.

part of the answer to part (iii)

$$\nabla^2 f(1, 1) = \begin{pmatrix} 12 & -4 \\ -14 & 12 \end{pmatrix};$$

Similarly, trace $\nabla^2 f(1, 1) > 0$ and $|\nabla^2 f(1, 1,)| > 0$,
thus $\nabla^2 f(1, 1)$ has 2 stictly positive eigenvalues and we saw that it is also the case with $\nabla^2 f(-1, 1)$.
$f(1, 1) = f(-1, -1) = -2 < 0$
Thus, the square $[-\sqrt{2}, \sqrt{2}] \times [-\sqrt{2}, \sqrt{2}]$ admits at least a minimum of $f(x, y)$, as it is compact.
Therefore $(1, 1)$ and $(-1, -1)$ are the only 2 minima of $f(x, y)$ in C and thus both are global minima in the square for $f(x, y)$.

Fig.(1) shows the curve $x^6 + y^6 = 1$. It is required to locate the point $P\,(x^*, y^*)$, on the curve, which is closest to the origin. Prove that $P\,(x^*, y^*)$ exists and explain how it is calculated.

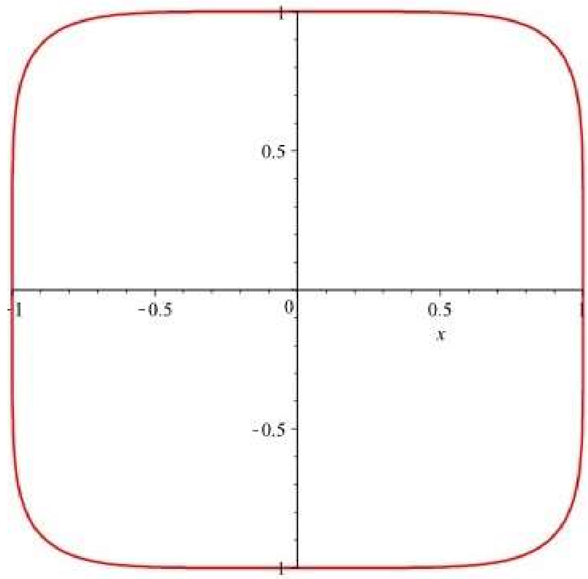


Figure 1: graph of $x^6 + y^6 = 1$

Short course on Multidisciplinary optimization and industrial applications

Optimisation - Introductory Examples

Prof MZ Dauhoo and Prof L Dumas

1 Optimisation

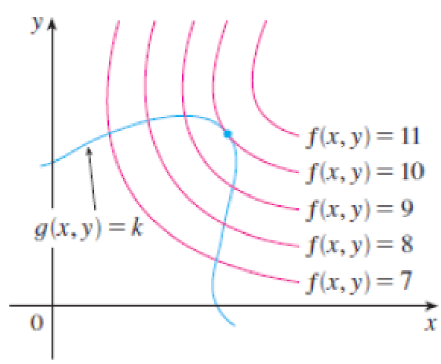
1.1 Unconstrained Optimisation

Let (x_0, y_0) be a critical point of a function, f which is twice continuously differentiable in $\mathcal{R}^2$. The nature of the critical point can be determined as follows:

$\det(H_f(x_0, y_0))$	$\partial_{xx} f(x_0, y_0)$	Nature of (x_0, y_0)
$+$	$+$	minimum
$+$	$-$	maximum
$-$		saddle point
0		no conlcusion

1.2 Constrained Optimisation

We are looking for the extrema of the function $f(x, y)$ under the constraint $g(x, y) = k$. This means that we are looking for the extrema of $f(x, y)$ when the point (x, y) belongs to the contour line $g(x, y) = k$. In Figure (1.2), we see this curve as well as several contour lines of f. These have the equation $f(x, y) = c$ for $c = 7, 8, 9, 10, 11$.



Optimise $f(x, y)$ under the condition $g(x, y) = k$ means to find the largest (or smallest) value of c such that the contour line $f(x, y) = c$ intersects the curve $g(x, y) = k$.

For this to take place, the two curves must have the same tangent line. This means that the gradients are parallel, that is, there exists $\lambda \in \mathcal{R}$ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$, where λ is the LAGRANGE multiplier.

Optimise $f(x, y)$ under the constraint $g(x, y)$.

We construct the Lagrangian as follows:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y), \text{where } \lambda \text{ is the Lagrange multiplier.}$$

For the function f to exhibit extremum, we require $\nabla \mathcal{L} = 0$.

$$\nabla \mathcal{L} = \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial x} \\ \\ \dfrac{\partial \mathcal{L}}{\partial y} \\ \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

Solving the above system, we get (x_0, y_0, λ_0) as the critical point.

In order to determine the nature of the critical point, we calculate the Hessian of the Lagrangian and determinine its nature as follows:

Let (x_0, y_0, λ_0) be a critical point of a function, f which is twice continuously differentiable in $\mathcal{R}^2$. The nature of the critical point can be determined as follows:

$\det(H_{\mathcal{L}}(x_0, y_0, \lambda_0))$	$\partial_{xx}\mathcal{L}(x_0, y_0, \lambda_0)$	Nature of (x_0, y_0, λ_0)
$+$	$+$	minimum
$+$	$-$	maximum
≤ 0		no conlcusion

Example 1:

Find the extrema and the corresponding nature of the function $f(x, y) = 5x^2 + 6y^2 - xy$ under the constraint $x + 2y = 24$.

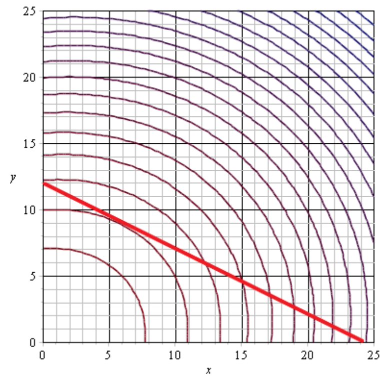


Figure 1: Sketch of the function $f(x, y)$ and the constraint.

We construct the Lagrangian as follows:

$$\begin{aligned} \mathcal{L}(x, y, \lambda) &= f(x, y) - \lambda g(x, y), \\ &= 5x^2 + 6y^2 - xy - \lambda(x + 2y - 24). \end{aligned} \tag{2}$$

For the function f to exhibit extremum, we require $\nabla \mathcal{L} = 0$.

$$\nabla \mathcal{L} = \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial x} \\ \\ \dfrac{\partial \mathcal{L}}{\partial y} \\ \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} 10x - y - \lambda \\ 12y - x - 2\lambda \\ -x - 2y + 24 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

Solving the above system, we get

$$x = 6, \quad y = 9 \quad \text{and} \quad \lambda = +51.$$

In order to determine the nature of the critical point, we calculate the Hessian of the Lagrangian as follows:

$$H_{\mathcal{L}}(x, y) = \begin{pmatrix} \dfrac{\partial^2 \mathcal{L}}{\partial x^2} & \dfrac{\partial^2 \mathcal{L}}{\partial x \partial y} \\ \\ \dfrac{\partial^2 \mathcal{L}}{\partial x \partial y} & \dfrac{\partial^2 \mathcal{L}}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ \\ -1 & 12 \end{pmatrix} \tag{4}$$

Since $\det(H_{\mathcal{L}}(x, y)) = 119 > 0$ and $\dfrac{\partial^2 \mathcal{L}}{\partial x^2} = 10 > 0$, therefore the critical point $(6, 9)$ is a minimum.