Laurent DUMAS, Versailles University

TD1: descent methods

Question 1
The following function is to be minimized on $\mathbb{R}^3$:

$$J(x, y, z) = x^2 + 2y^2 + z^2 - x + y - z.$$

a Give the exact solution of the problem considered, explaining your answer clearly.

b The gradient descent method, with a backtracking strategy using Armijo condition is used in order to find the numerical solution using the initial approximation $X_0 = (0, 1, 1)$.

(i) What is the descent direction at the first iteration? *Recall that the condition of Armijo for a starting point X_0 and a direction of descent d is given by:*

$$J(X_0 + \alpha d) \leq J(X_0) + \beta\alpha\langle d, \nabla J(X_0)\rangle$$

(ii) For the function J, defined above, calculate explicitly and represent graphically in this case the function $\alpha \mapsto J(X_0 + \alpha d)$. Find the corresponding values of α satisfying the Armijo condition for $\beta = 0.1$.

c Find X_1 for $\alpha_{init} = 1$ and $\tau = 0.1$, the backtracking constant.

Question 2

Consider the function $f \in \mathcal{C}^1(\mathbb{R})$, such that $\lim_{\|x\| \to +\infty} f(x) = +\infty$. Let g be the function gradient of $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. g is assumed to be Lipschitzian on the set

a Show that f has a global minimum x^* for which $g(x^*) = 0$.

b The aim is to find a convergent sequence $(x_k)_{k \in N}$, so that its limit is a minimum (local or global) of f.
Given x_0 and the following relation:

$$x_{k+1} = x_k + t_k d_k,$$

where d_k is the descent direction satisfying $(d_k, g(x_k))$, t_k is the step in that direction, satisfying the following inequalities:

$$q(t_k) \leq q(0) + m_1 t_k q'(0) \quad \text{and} \quad q'(t_k) \geq m_2 q'(0),$$

$q(t) = f(x_k + t d_k)$ and m_1 ann m_2 satisfy the following inequality:

$$0 < m_1 < m_2 < 1.$$

Sketch the graph of a function of one variable and indicate the admissible values of t_k on the graph.

c It is required to show that the method of descent described above, converges towards a critical point of f when $d_k = -g(x_k) = -g_k$ (gradient direction).

(i) Show that in this case, $q'(0) = -\|g_k\|^2$.

(ii) Show that $m_1 \|g_k\| \cdot \|x_{k+1} - x_k\| \leq f(x_k) - f(x_{k+1})$.

(iii) Show that $(1 - m_2)||g_k|| \leq L||x_{k+1} - x_k||$ where L is the Lipschitz constant of g in S_{x_0}.

(iv) Deduce that $\lim_{k \to +\infty} g_k = 0$.
Optional: Suggest an algorithm which can generate the sequence x_k, and thus determine the correct $t_k \in \mathbb{R}$.

Question 3

a Graph a few contours of the function given by:

$$f(x, y) = 2(x - 1)^2 + y^2 + 1$$

then at a given point one of the contours, draw the gradient and a descent direction.

b Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be differentiable and $x \in \mathbb{R}^n$. It is assumed that $d \in \mathbb{R}$ is such that

$$\|f(x) + d\| \leq \|\nabla f(x)\|.$$

Show that d is a direction of descent of f at x.

c Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be differentiable and convex. Let x and y be elements of $\mathbb{R}^n$, such as $f(y) < f(x)$. Show that $y - x$ is a direction of descent of f at x.

Question 4

It is required to minimize the following function:

$$J(x, y, z) = x^4 + 2y^4 + z^4 - 2x + y - z$$

on $\mathbb{R}^3$.

a Give the exact solution of the above minimization problem.

b Newton's method is used with a step $\alpha = 1$ in order to obtain a numerical solution using the initial approximation $X_0 = (1, 1, 1)$. What is the direction of the descent direction at the first iteration? verify that it is effectively a descent direction.

c Find X_1.