DERVATIVE FREE OPTIMIZATION
FINAL EXAM, PART 2

Exercice 1 *On the Nelder Mead algorithm*

Consider the Nelder Mead algorithm for the sphere function in $\mathbb{R}^2$:

$$f(x,y) = x^2 + y^2$$

and the initial simplex made of $A = (4,5)$, $B = (5,3)$ and $C = (5,6)$.

Compute the best obtained value by this algorithm after one iteration.

Exercice 2 *On the Cauchy Step in a trust region method*

Consider the following quadratic model in the closed ball $B(x_0, R)$ of $\mathbb{R}^n$:

$$m(x_0 + h) = c + < g, h > + \frac{1}{2}{}^t hHh$$

where $c \in \mathbb{R}$, $g \in (\mathbb{R}^n)^*$ and $H \in \mathcal{M}_n(\mathbb{R})$.

We denote by t_c the Cauchy step in the steepest descent direction, that is :

$$t_c = \text{argmin}\{m(x_0 - tg), t > 0, x_0 - tg \in B(x_0, R)\}$$

The aim is to prove that

$$m(x_0) - m(x_0 - t_c g) \geq \frac{1}{2}||g|| \min(\frac{||g||}{|||H|||}, R)$$

where $\frac{||g||}{|||H|||} = +\infty$ if $H = 0$.

1. First, we assume that ${}^t gHg > 0$. Prove that in this case

   $$t_c = \min(\frac{R}{||g||}, \frac{||g||^2}{{}^t gHg})$$

   and conclude
2. Assume now that ${}^g Hg \leq 0$. Prove that $t_c = \frac{R}{||g||}$ in this case and conclude.

Exercice 3 *On the Kriging model*

The Kriging model is a surrogate model of a given function $J : \mathbb{R}^n \to \mathbb{R}$ that can be written as :

$$\hat{J}(X) = \sum_{i=1}^{N} \omega(X_i) J(X_i)$$

where the N points $X_i \in \mathbb{R}^n$ have been previously evaluated by J.

In this expression, J and $\hat{J}$ are viewed as a random fields where J is assumed to have a zero mean value everywhere. Moreover, the covariance between the evaluation at two points X and Y has the following form :

$$cov(J(X), J(Y)) = c(X, Y)$$

where the function $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is supposed to be known.

At each point $X \in \mathbb{R}^n$, the random field $\hat{J}$ is defined so as to minimize the standard deviation between $J(X)$ and $\hat{J}(X)$, while ensuring $E(\hat{J}(X)) = E(J(X))$.

1. Prove that
$$\hat{J}(X) =^t KC^{-1} z$$
where $K =^t (c(X_1, X), ..., c(X_N, X))$, $z =^t (J(X_1), ...J(X_N))$ and C is a $N \times N$ matrix such that $C_{i,j} = c(X_i, X_j)$.
2. Verify that the Kriging function is an interpolation model, that is $\hat{J}(X_i) = J(X_i)$ for all $i \in \{1, ..., N\}$.
3. Give an expression of $Var(J(X) - \hat{J}(X))$ using $c(X, X)$, K and C.